SPT-05/011, UB-ECM-PF 05/03

2-matrix versus complex matrix model, integrals over the unitary group as triangular integrals

B. Eynard[1]

Service de Physique Théorique de Saclay, CEA/DSM/SPhT - CNRS/SPM/URA 2306,
F-91191 Gif-sur-Yvette Cedex, France.

A. Prats Ferrer[2]

Universitat Barcelona, Departament d'Estructura i constituents de la Matèria.
Av. Diagonal 647, 08028, Barcelona.

Abstract

We prove that the 2-hermitean matrix model and the complex-matrix model obey the same loop equations, and as a byproduct, we find a formula for Itzykzon-Zuber's type integrals over the unitary group. Integrals over $U(n)$ are rewritten as gaussian integrals over triangular matrices and then computed explicitely. That formula is an efficient alternative to the former Shatashvili's formula.

[1]E-mail: bertrand.eynard@cea.fr
[2]E-mail: prats@ecm.ub.es

1 Introduction

It has been noticed for a long time now, that the so called "Two-Hermitean-Matrix-Model" (introduced in particular for quantum gravity [19, 10]) and the so called "Complex-Matrix-Model" (used in particular for its applications to Laplacian growth models [27, 28], and string theory [1]) share lots of similarities: They have the same leading large N expansion properties, and, both are associated to some ensembles of biorthogonal polynomials which have formaly the same properties. Here, we add a new piece to make this correspondence more precise, we prove that both models have the same loop equations.

Both models are not defined for the same weights, in fact, the set of weights for which one model is well defined has no intersection with the set of weights for which the other model is well defined. However, each model can be analyticaly continued to a larger set of weights, and in that sense, the two models coincide.

When written in terms of eigenvalues, this identification of the 2-hermitean-matrix-model and complex-matrix-model has some interesting corolary: it gives a formula for computing integrals (of the Itzykzon-Zuber type) over the unitary group, as gaussian integrals over triangular matrices. Therefore, we obtain a very explicit formula for all correlators of the Shatashvili's type [24]. In [24] S. Shatashvili found a formula for all $U(n)$ correlation functions, but his formula still contains integrals, is not explicitely symmetric in all variables, and is very difficult to use for practical purposes, such as [6]. In the particular case of the 2-point correlation function, Morozov has found a much simpler formula [23]. In [23] A. Morozov computed it for $U(n)$ with $n \leq 3$ and conjectured it for $n > 3$. Morozov's formula was later proven for all n in [6], and written in an even simpler form [13]. Here, we find a natural generalization of Morozov's formula. The formula we find here, contains no integration, it gives the $U(n)$ correlation functions as the sum of a finite number of terms, and is very efficient for effective computations. It also provides an alternative new proof of Itzykzon-Zuber's formula.

The derivations proposed in this article are elementary, and it would be interesting to put them in the more general framework of group representation theory [20, 17].

The main results presented in this paper are:

• Theorem 3.3 and in particular Remark 3.3, which states the equivalence between the Hermitean-2-matrix model and the complex-matrix model:

$$\int_{H_n \times H_n} dM_1\, dM_2\, F(M_1, M_2)\, \mathrm{e}^{-\gamma \operatorname{Tr} M_1 M_2} \equiv \int_{GL_n(\mathrm{C})} dZ\, F(Z, Z^\dagger)\, \mathrm{e}^{-\gamma \operatorname{Tr} Z Z^\dagger} \tag{1-1}$$

The definitions of each terms and the meaning of that equality are explained in section 3.3.

• Theorem 4.1, which allows to compute $U(n)$ integrals as triangular integrals.

$$\int_{U(n)} dU\, F(X, UYU^\dagger)\, \mathrm{e}^{-\operatorname{Tr} XUYU^\dagger}$$

$$\propto \frac{\sum_\sigma \sum_\tau (-1)^\sigma (-1)^\tau \mathrm{e}^{-\,\mathrm{Tr}\, X_\sigma Y_\tau} \int_{T(n)} F(X_\sigma + T, Y_\tau + T^\dagger)\, \mathrm{e}^{-\,\mathrm{Tr}\, TT^\dagger}}{\Delta(X)\Delta(Y)} \tag{1-2}$$

for any polynomial invariant function F.

• Theorem 5.2, which gives a formula for computing triangular matrix gaussian integrals. We parametrize polynomial invariant functions by pairs of permutations (of some size R), and a basis is written $F_{\pi,\pi'}$. Theorem 5.2 gives the result of integration over triangular matrices:

$$\begin{aligned} & \frac{\int_{T(n)} dT\, \mathrm{e}^{-\,\mathrm{Tr}\, TT^\dagger}\, F_{\pi,\pi'}(\vec{x},\vec{y},X+T,Y+T^\dagger)}{\int_{T(n)} dT\, \mathrm{e}^{-\,\mathrm{Tr}\, TT^\dagger}} \\ = & \left(\mathcal{M}^{(R)}(\vec{x},\vec{y},X_n,Y_n)\,\mathcal{M}^{(R)}(\vec{x},\vec{y},X_{n-1},Y_{n-1})\,\ldots\mathcal{M}^{(R)}(\vec{x},\vec{y},X_1,Y_1)\right)_{\pi,\pi'} \end{aligned} \tag{1-3}$$

where $\mathcal{M}^{(R)}(\vec{x},\vec{y},X_n,Y_n)$ is the matrix of size $R!$, indexed by pairs of permutations:

$$\mathcal{M}^{(R)}_{\pi,\rho}(\vec{x},\vec{y},X_n,Y_n) = \prod_{i=1}^{R}\left(\delta_{\pi(i),\rho(i)} + \frac{1}{(x_i - X_n)(y_{\pi(i)} - Y_n)}\right) \tag{1-4}$$

Theorem 5.3 shows that the matrices in eq.1-3 commute together, and can be simultaneousy diagonalized.

• Theorem 6.1, which gives a formula for computing correlation functions in terms of biorthogonal polynomials:

$$\frac{\int_{H_n\times H_n} dM_1\, dM_2\, F_{\pi,\pi'}(\vec{x},\vec{y},M_1,M_2)\, \mathrm{e}^{-\,\mathrm{Tr}\,(V_1(M_1)+V_2(M_2)+M_1M_2)}}{\int_{H_n\times H_n} dM_1\, dM_2\, \mathrm{e}^{-\,\mathrm{Tr}\,(V_1(M_1)+V_2(M_2)+\gamma M_1 M_2)}} = \left(\mathrm{Mdet}\left(\mathcal{M}^{(R)}(\vec{x},\vec{y},Q,P^t)\right)\right)_{\pi,\pi'} \tag{1-5}$$

where notations are explained in section 6.2.

Outline:

• In part 2 we give definitions of groups and measures.

• In part 3, we prove the equivalence between the Hermitean-2-matrix model and the complex-matrix model, by showing that they have the same loop equations.

• In part 4, we prove the identity between $U(n)$ integrals and triangular integrals, and give some examples. In particular we rederive Itzykson-Zuber's formula and Morozov's formula.

• In part 5, we compute the triangular integrals, by parametrizing polynomial invariant functions with pairs of permutations. In particular we explicit all four point functions.

• In part 6, we integrate over eigenvalues using biorthogonal polynomials technics, and get expressions for correlation functions.

2 Definitions

2.1 Ensembles

Let

- $U(n)$:= group of $n \times n$ unitary matrices, with the normalized Haar measure.
- H_n := group of $n \times n$ hermitean matrices, with the Lebesgue measure:

$$dM := \prod_i dM_{ii} \prod_{i<j} d\mathrm{Re}M_{ij}\, d\mathrm{Im}M_{ij} \tag{2-1}$$

- $GL_n(\mathrm{C})$:= group of $n \times n$ complex matrices, with the Lebesgue measure:

$$dZ := \prod_{i,j} d\mathrm{Re}Z_{ij}\, d\mathrm{Im}Z_{ij} \tag{2-2}$$

- T_n := group of $n \times n$ strictly upper triangular complex matrices, with the Lebesgue measure:

$$dT := \prod_{i<j} d\mathrm{Re}T_{ij}\, d\mathrm{Im}T_{ij} \tag{2-3}$$

- $D_n(\mathrm{R})$:= group of $n \times n$ real diagonal matrices, with the Lebesgue measure:

$$dX := \prod_i dX_{ii} \tag{2-4}$$

- $D_n(\mathrm{C})$:= group of $n \times n$ complex diagonal matrices, with the Lebesgue measure:

$$dX := \prod_i d\mathrm{Re}X_{ii}\, d\mathrm{Im}X_{ii} \tag{2-5}$$

- $\Sigma(n)$:= group of permutations of n elements.

2.2 Vandermonde determinant

For any diagonal matrix $X = \mathrm{diag}(X_1, \dots, X_n) \in D_n(\mathrm{C})$, one writes:

$$\Delta(X) := \prod_{i<j} (X_i - X_j) \tag{2-6}$$

and, for any permutation $\sigma \in \Sigma(n)$, we define the diagonal matrix:

$$X_\sigma := \mathrm{diag}(X_{\sigma(1)}, \dots, X_{\sigma(n)}) \tag{2-7}$$

Notice that:

$$\Delta(X_\sigma) := (-1)^\sigma\ \Delta(X) \tag{2-8}$$

2.3 Invariant functions

Definition 2.1 *$F(A,B)$ defined on $GL_n(\mathrm{C}) \times GL_n(\mathrm{C}) \to \mathrm{C}$ is an analytical invariant function if:*

- *F is analytical in each variable,*
- *$\forall U \in GL_n^*(\mathrm{C}), F(UAU^{-1}, UBU^{-1}) = F(A,B)$.*

Examples:

$$F(A,B) = \prod_{t=1}^{p} \operatorname{Tr} \left(\prod_{r_t=1}^{R_t} (x_{t,r_t} - A)(y_{t,r_t} - B) \right) \tag{2-9}$$

$$F(A,B) = \mathrm{e}^{-\operatorname{Tr} V_1(A)}\, \mathrm{e}^{-\operatorname{Tr} V_2(B)} \tag{2-10}$$

Definition 2.2 *Monomial invariant functions are functions of the form:*

$$F(A,B) = \prod_{t=1}^{p} \operatorname{Tr} \left(\prod_{r_t=1}^{R_t} (A^{k_{t,r_t}}\, B^{l_{t,r_t}}) \right) \tag{2-11}$$

where the k_{t,r_t}'s and l_{t,r_t}'s are integers such that $k_{t,r_t} + l_{t,r_t} > 0$. The total degree is

$$\deg F := \sum_{t=1}^{p} \sum_{r_t=1}^{R_t} k_{t,r_t} + l_{t,r_t} \tag{2-12}$$

Definition 2.3 *Polynomial invariant functions are finite complex linear combinations of monomial invariant functions.*

Examples of polynomial invariant functions:

$$F(A,B) = \operatorname{Tr} A^{k_1}\, B^{l_1}\, A^{k_2}\, B^{l_2} \quad , \qquad F(A,B) = \left(1 + \operatorname{Tr} A^{k_1}\, B^{l_1}\right) \left(1 + \operatorname{Tr} A^{k_2}\, B^{l_2}\right) \tag{2-13}$$

$$F(A,B) = \prod_{t=1}^{p} \det(x_t - A)^{k_t} \prod_{u=1}^{q} \det(y_u - B)^{l_u} \tag{2-14}$$

$$F(A,B) = \det(A \otimes 1 - 1 \otimes B) \tag{2-15}$$

2.4 Decompositions

2.4.1 Diagonalization

It is a standard result in algebra (see [22, 17, 20] for instance), that any hermitean matrix $M \in H_n$ can be written:

$$M = UXU^{\dagger} \tag{2-16}$$

where $U \in U(n)$ and $X \in D_n(\mathrm{R})$.

The measure is then:

$$dM = \tilde{J}_n \, \Delta^2(X) \, dU \, dX \tag{2-17}$$

where the Jacobian is

$$\tilde{J}_n = \frac{\pi^{\frac{n(n-1)}{2}}}{\prod_{k=0}^{n-1} k!} \tag{2-18}$$

This decomposition is not unique. It is unique up to a permutation of eigenvalues, and up to multiplication of U by a diagonal matrix whose elements are on the unit circle. In other words, $M = UXU^\dagger$ provides a mapping between H_n and $U(n) \times D_n(\mathrm{R})/(U(1)^n \times \Sigma(n))$.

2.4.2 Jordanization

A less standard result (see [22, 26, 20, 17] for instance), is that any complex matrix $Z \in GL_n(\mathrm{C})$ can be written:

$$Z = U(X + T)U^\dagger \tag{2-19}$$

where $U \in U(n)$, $T \in T_n$ and $X \in D_n(\mathrm{C})$.

The measure is then:

$$dZ = J_n \, |\Delta(X)|^2 \, dU \, dT \, dX \tag{2-20}$$

where the Jacobian is

$$J_n = \frac{\left(\frac{\pi}{2}\right)^{\frac{n(n-1)}{2}}}{\prod_{k=0}^{n-1} k!} \tag{2-21}$$

This decomposition is not unique. It is unique up to a permutation of eigenvalues, and up to multiplication of U by a diagonal matrix whose elements are on the unit circle. In other words, $Z = U(X + T)U^\dagger$ provides a mapping between $Gl_n(\mathrm{C})$ and $U(n) \times T_n \times D_n(\mathrm{C})/(U(1)^n \times \Sigma(n))$.

3 Gaussian matrix integrals

In all what follows, we consider 3 complex numbers α_1, α_2 and γ, and we define

$$\delta := \alpha_1\alpha_2 - \gamma^2 \tag{3-1}$$

and assume that $\delta \neq 0$.

3.1 Gaussian Hermitean model

Consider the measure on $H_n \times H_n$:

$$\mathrm{e}^{-\operatorname{Tr}\left(\frac{\alpha_1}{2}M_1^2 + \frac{\alpha_2}{2}M_2^2 + \gamma M_1 M_2\right)} \, dM_1 \, dM_2 \tag{3-2}$$

Definition 3.1 *The partition function is:*

$$Z_H(n,\gamma,\alpha_1,\alpha_2) := \int_{H_n\times H_n} dM_1\, dM_2\, \mathrm{e}^{-\,\mathrm{Tr}\,(\frac{\alpha_1}{2}M_1^2+\frac{\alpha_2}{2}M_2^2+\gamma M_1 M_2)} \tag{3-3}$$

Notice that the integral Z_H is absolutely convergent only if

$$\forall \phi\in \mathrm{R} \qquad \mathrm{Re}(\alpha_1 \mathrm{e}^{\phi}+\alpha_2\mathrm{e}^{-\phi}\pm 2\gamma)>0 \tag{3-4}$$

which implies that $\mathrm{Re}\alpha_1>0$, $\mathrm{Re}\alpha_2>0$, $(\mathrm{Re}\gamma)^2<\mathrm{Re}\alpha_1\mathrm{Re}\alpha_2$.

An easy gaussian integral computation gives:

$$Z_H = 2^n \left(\frac{\pi}{\sqrt{\delta}}\right)^{n^2} \quad . \tag{3-5}$$

Definition 3.2 *The expectation value of an invariant function $F(A,B)$ is:*

$$\langle F\rangle_H := \frac{\int_{H_n\times H_n} dM_1\, dM_2\, F(M_1,M_2)\, \mathrm{e}^{-\,\mathrm{Tr}\,(\frac{\alpha_1}{2}M_1^2+\frac{\alpha_2}{2}M_2^2+\gamma M_1 M_2)}}{\int_{H_n\times H_n} dM_1\, dM_2\, \mathrm{e}^{-\,\mathrm{Tr}\,(\frac{\alpha_1}{2}M_1^2+\frac{\alpha_2}{2}M_2^2+\gamma M_1 M_2)}} \tag{3-6}$$

Remark 3.1 It is clear, from Wick's theorem, that if F is a monomial invariant function, then $<F>_H$ is a polynomial in $\frac{\alpha_1}{\delta}$, $\frac{\alpha_2}{\delta}$ and $\frac{\gamma}{\delta}$, and can be analiticaly continued to every complex α_1,α_2,γ, provided that $\delta\neq 0$.

3.1.1 Gaussian Hermitean loop equations

Consider a monomial matrix valued function, of the form:

$$f(A,B)=f_0(A,B)\prod_{t=1}^{p} \mathrm{Tr}\, f_t(A,B)\ , \qquad \forall t=0,\dots,p, \quad f_t(A,B)=\prod_{r_t=1}^{R_t} A^{k_{t,r_t}}B^{l_{t,r_t}} \tag{3-7}$$

define:

$$G_0(A,B):=\prod_{u\neq 0}\mathrm{Tr}\, f_u(A,B)\ , \qquad \text{and if } t\geq 1\,,\ G_t(A,B):=\prod_{u\neq 0,t}\mathrm{Tr}\, f_u(A,B) \tag{3-8}$$

Theorem 3.1 *One has the "loop equations":*

$$\begin{aligned}
& \alpha_1 \left\langle G_0(M_1,M_2)\,\mathrm{Tr}\, M_1 f_0(M_1,M_2)\right\rangle_H + \gamma\left\langle G_0(M_1,M_2)\,\mathrm{Tr}\, M_2 f_0(M_1,M_2))\right\rangle_H \\
= \quad & \sum_{r=1}^{R_0}\sum_{j=0}^{k_{0,r}-1}\Bigg\langle G_0(M_1,M_2)\ \mathrm{Tr}\left(\left(\prod_{u=1}^{r-1} M_1^{k_{0,u}}{M_2}^{l_{0,u}}\right)M_1^j\right) \\
& \qquad\qquad \mathrm{Tr}\left(M_1^{k_{0,r}-j-1}{M_2}^{l_{0,r}}\left(\prod_{u=r+1}^{R_0} M_1^{k_{0,u}}{M_2}^{l_{0,u}}\right)\right)\Bigg\rangle_H \\
& +\sum_{t=1}^{p}\sum_{r=1}^{R_t}\sum_{j=0}^{k_{t,r}-1}\Bigg\langle G_t(M_1,M_2)\ \mathrm{Tr}\left(\left(\prod_{u=1}^{r-1} M_1^{k_{t,u}}{M_2}^{l_{t,u}}\right)M_1^j f_0(M_1,M_2) M_1^{k_{t,r}-j-1}{M_2}^{l_{t,r}}\right.
\end{aligned}$$

$$\left(\prod_{u=r+1}^{R_t} M_1^{k_{t,u}} {M_2}^{l_{t,u}}\right)\right)\right\rangle_H$$

(3 − 9)

and

$$\begin{aligned} & \alpha_2 \left\langle G_0(M_1,M_2)\,\mathrm{Tr}\,M_2 f_0(M_1,M_2)\right\rangle_H + \gamma\left\langle G_0(M_1,M_2)\,\mathrm{Tr}\,M_1 f_0(M_1,M_2))\right\rangle_H \\ = \quad & \sum_{r=1}^{R_0}\sum_{j=0}^{l_{0,r}-1} \left\langle G_0(M_1,M_2)\ \mathrm{Tr}\ \left(\left(\prod_{u=1}^{r-1} M_1^{k_{0,u}} {M_2}^{l_{0,u}}\right) M_1^{k_{0,r}} {M_2}^{j}\right)\right. \\ & \qquad \left.\mathrm{Tr}\ \left({M_2}^{l_{0,r}-j-1}\left(\prod_{u=r+1}^{R_0} M_1^{k_{0,u}} {M_2}^{l_{0,u}}\right)\right)\right\rangle_H \\ & + \sum_{t=1}^{p}\sum_{r=1}^{R_t}\sum_{j=0}^{l_{t,r}-1} \left\langle G_t(M_1,M_2)\ \mathrm{Tr}\ \left(\left(\prod_{u=1}^{r-1} M_1^{k_{t,u}} {M_2}^{l_{t,u}}\right) {M_1}^{k_{t,r}} {M_2}^{j} f_0(M_1,M_2) {M_2}^{l_{t,r}-j-1}\right.\right. \\ & \qquad \left.\left.\left(\prod_{u=r+1}^{R_t} M_1^{k_{t,u}} {M_2}^{l_{t,u}}\right)\right)\right\rangle_H \end{aligned}$$

(3 − 10)

Notice that the RHS is a linear combination of invariant polynomial functions of degree strictly lower than the LHS.

Loop equations are a standard method for finding recursion relations among expectation values [10], they were first studied by [25] for the 2-matrix model, and solved more explicitely by [15, 12, 14].

proof:

Write that the integral of a total derivative is zero:

$$0 = \sum_i \int dM_1\, dM_2\, \frac{\partial}{\partial M_{1ii}}\ \left(f_{i,i}(M_1,M_2)\,\mathrm{e}^{-\,\mathrm{Tr}\,(\frac{\alpha_1}{2}M_1^2+\frac{\alpha_2}{2}M_2^2+\gamma M_1 M_2)}\right) \tag{3-11}$$

i.e.

$$\begin{aligned} & \sum_i \int dM_1\, dM_2\, \left(\frac{\partial}{\partial M_{1ii}}\ f_{i,i}(M_1,M_2)\right)\ \mathrm{e}^{-\,\mathrm{Tr}\,(\frac{\alpha_1}{2}M_1^2+\frac{\alpha_2}{2}M_2^2+\gamma M_1 M_2)} \\ = \quad & \sum_i \int dM_1\, dM_2\, f_{i,i}(M_1,M_2)\ (\alpha_1 M_{1ii} + \gamma M_{2ii})\ \mathrm{e}^{-\,\mathrm{Tr}\,(\frac{\alpha_1}{2}M_1^2+\frac{\alpha_2}{2}M_2^2+\gamma M_1 M_2)} \end{aligned}$$

(3 − 12)

Similarly:

$$\begin{aligned} & \sum_{i<j} \int dM_1\, dM_2\, \left(\frac{\partial}{\partial \mathrm{Re} M_{1ij}}\ f_{i,j}(M_1,M_2)\right)\ \mathrm{e}^{-\,\mathrm{Tr}\,(\frac{\alpha_1}{2}M_1^2+\frac{\alpha_2}{2}M_2^2+\gamma M_1 M_2)} \\ = \quad & \sum_{i<j} \int dM_1\, dM_2\, f_{i,j}(M_1,M_2)\ \big(\alpha_1 (M_{1ji}+M_{1ij}) + \gamma (M_{2ji}+M_{2ij})\big) \end{aligned}$$

$$\mathrm{e}^{-\operatorname{Tr}\left(\frac{\alpha_1}{2}M_1^2+\frac{\alpha_2}{2}M_2^2+\gamma M_1M_2\right)}$$

$(3-13)$

and

$$\sum_{i<j}\int dM_1\,dM_2\,\left(\frac{\partial}{\partial \mathrm{Im} M_{1ij}}\,f_{i,j}(M_1,M_2)\right)\,\mathrm{e}^{-\operatorname{Tr}\left(\frac{\alpha_1}{2}M_1^2+\frac{\alpha_2}{2}M_2^2+\gamma M_1M_2\right)}$$
$$= \quad i\sum_{i<j}\int dM_1\,dM_2\,f_{i,j}(M_1,M_2)\,\left(\alpha_1(M_{1ji}-M_{1ij})+\gamma(M_{2ji}-M_{2ij})\right)$$
$$\mathrm{e}^{-\operatorname{Tr}\left(\frac{\alpha_1}{2}M_1^2+\frac{\alpha_2}{2}M_2^2+\gamma M_1M_2\right)}$$

$(3-14)$

Taking 3-12 + 3-13 $-i$ 3-14 , we get:

$$\begin{aligned}
&\sum_{i}\int dM_1\,dM_2\,\left(\frac{\partial}{\partial M_{1ii}}\,f_{i,i}(M_1,M_2)\right)\,\mathrm{e}^{-\operatorname{Tr}\left(\frac{\alpha_1}{2}M_1^2+\frac{\alpha_2}{2}M_2^2+\gamma M_1M_2\right)}\\
&+\frac{1}{2}\sum_{i<j}\int dM_1\,dM_2\,\left(\left(\frac{\partial}{\partial \mathrm{Re} M_{1ij}}-i\frac{\partial}{\partial \mathrm{Im} M_{1ij}}\right)\,f_{i,j}(M_1,M_2)\right)\,\mathrm{e}^{-\operatorname{Tr}\left(\frac{\alpha_1}{2}M_1^2+\frac{\alpha_2}{2}M_2^2+\gamma M_1M_2\right)}\\
&+\frac{1}{2}\sum_{i<j}\int dM_1\,dM_2\,\left(\left(\frac{\partial}{\partial \mathrm{Re} M_{1ij}}+i\frac{\partial}{\partial \mathrm{Im} M_{1ij}}\right)\,f_{j,i}(M_1,M_2)\right)\,\mathrm{e}^{-\operatorname{Tr}\left(\frac{\alpha_1}{2}M_1^2+\frac{\alpha_2}{2}M_2^2+\gamma M_1M_2\right)}\\
&=\int dM_1\,dM_2\,\left(\operatorname{Tr} f(M_1,M_2)(\alpha_1M_1+\gamma M_2)\right)\,\mathrm{e}^{-\operatorname{Tr}\left(\frac{\alpha_1}{2}M_1^2+\frac{\alpha_2}{2}M_2^2+\gamma M_1M_2\right)}
\end{aligned} \tag{3-15}$$

i.e. one can proceed as if all the M_{1ij} were n^2 real indepedent variables, i.e., by abuse of notation we write:

$$\sum_{i,j}\int dM_1\,dM_2\,\left(\frac{\partial}{\partial M_{1ij}}\,f_{i,j}(M_1,M_2)\right)\,\mathrm{e}^{-\operatorname{Tr}\left(\frac{\alpha_1}{2}M_1^2+\frac{\alpha_2}{2}M_2^2+\gamma M_1M_2\right)}$$
$$= \quad \int dM_1\,dM_2\,\left(\operatorname{Tr} f(M_1,M_2)(\alpha_1M_1+\gamma M_2)\right)\,\mathrm{e}^{-\operatorname{Tr}\left(\frac{\alpha_1}{2}M_1^2+\frac{\alpha_2}{2}M_2^2+\gamma M_1M_2\right)}$$

$(3-16)$

Now, one can use the following rules:

- split rule: if $f(M_1,M_2)=AM_1^kB$ (where A and B are matrices), one has:

$$\sum_{i,j}\frac{\partial f(M_1,M_2)_{ij}}{\partial M_1 ij}=\sum_{l=0}^{k-1}\operatorname{Tr}\,\left(AM_1^{k-1-l}\right)\,\operatorname{Tr}\,\left(M_1^lB\right) \tag{3-17}$$

- merge rule: if $f(M_1,M_2)=A\operatorname{Tr}(M_1^kB)$ (where A and B are matrices), one has:

$$\sum_{i,j}\frac{\partial f(M_1,M_2)_{ij}}{\partial M_1 ij}=\sum_{l=0}^{k-1}\operatorname{Tr}\,\left(AM_1^{k-1-l}BM_1^l\right) \tag{3-18}$$

Then, if A and B depend on M_1, one has to use the chain rule.

When one considers f given by 3-7, one gets eq.3-9.

Eq.3-10 is obtained by doing the same for M_2. $\square$

We find again that $\langle F\rangle_H$ is a polynomial in $\frac{\alpha_1}{\delta}$, $\frac{\alpha_2}{\delta}$ and $\frac{\gamma}{\delta}$.

3.2 Gaussian Complex model

Consider the measure on $GL_n(\mathrm{C})$:

$$\mathrm{e}^{-\operatorname{Tr}\left(\frac{\alpha_1}{2}Z^2+\frac{\alpha_2}{2}Z^{\dagger 2}+\gamma ZZ^\dagger\right)}\,dZ \tag{3-19}$$

Definition 3.3 *The partition function is:*

$$Z_C(n,\gamma,\alpha_1,\alpha_2) := \int_{Gl_n(\mathrm{C})} dZ\,\mathrm{e}^{-\operatorname{Tr}\left(\frac{\alpha_1}{2}Z^2+\frac{\alpha_2}{2}Z^{\dagger 2}+\gamma ZZ^\dagger\right)} \tag{3-20}$$

Notice that the integral Z_C is absolutely convergent only if

$$\forall\theta\in\mathrm{R} \qquad \mathrm{Re}(\alpha_1\mathrm{e}^{i\theta}+\alpha_2\mathrm{e}^{-i\theta}+2\gamma)>0 \tag{3-21}$$

One can see that with $\theta=\pi$, this condition can never be compatible with 3-4 (with $\phi=0$). Therefore, if Z_H is an absolutely convergent integral then Z_C is not, and vice–versa.

An easy gaussian integration gives (where $\delta=\alpha_1\alpha_2-\gamma^2$):

$$Z_C = \left(\frac{\pi}{\sqrt{-\delta}}\right)^{n^2} \tag{3-22}$$

which can be analiticaly continued to every α_1,α_2,γ, provided that $\delta\neq 0$.

Definition 3.4 *The expectation value of an invariant function $F(A,B)$ is:*

$$\langle F\rangle_C := \frac{\int_{Gl_n(\mathrm{C})} dZ\,F(Z,Z^\dagger)\,\mathrm{e}^{-\operatorname{Tr}\left(\frac{\alpha_1}{2}Z^2+\frac{\alpha_2}{2}Z^{\dagger 2}+\gamma ZZ^\dagger\right)}}{\int_{Gl_n(\mathrm{C})} dZ\,\mathrm{e}^{-\operatorname{Tr}\left(\frac{\alpha_1}{2}Z^2+\frac{\alpha_2}{2}Z^{\dagger 2}+\gamma ZZ^\dagger\right)}} \tag{3-23}$$

Remark 3.2 It is clear, from Wick's theorem, that if F is a monomial invariant function, then $< F >_C$ is a polynomial in $\frac{\alpha_1}{\delta}$, $\frac{\alpha_2}{\delta}$ and $\frac{\gamma}{\delta}$, and can be analiticaly continued to every complex α_1,α_2,γ, provided that $\delta\neq 0$.

3.2.1 Gaussian complex loop equations

Consider a monomial matrix valued function, of the form:

$$f(Z,Z^\dagger)=f_0(Z,Z^\dagger)\prod_{t=1}^{p}\operatorname{Tr}f_t(Z,Z^\dagger)\ , \qquad f_t(Z,Z^\dagger)=\prod_{r_t=1}^{R_t} Z^{k_{t,r_t}}{Z^\dagger}^{l_{t,r_t}} \tag{3-24}$$

define:

$$G_0(Z,Z^\dagger) := \prod_{u\neq 0}\operatorname{Tr}f_u(Z,Z^\dagger)\ , \qquad \text{and if } t\geq 1\,,\ G_t(Z,Z^\dagger) := \prod_{u\neq 0,t}\operatorname{Tr}f_u(Z,Z^\dagger) \tag{3-25}$$

Theorem 3.2 *One has the same loop equations than theorem 3.1, with replacing the subscript H by C.*

$$\begin{aligned}
&\alpha_1 \left\langle G_0(Z,Z^\dagger)\,\mathrm{Tr}\, Z f_0(Z,Z^\dagger)\right\rangle_C + \gamma \left\langle G_0(Z,Z^\dagger)\,\mathrm{Tr}\, Z^\dagger f_0(Z,Z^\dagger))\right\rangle_C \\
= \quad & \sum_{r=1}^{R_0}\sum_{j=0}^{k_{0,r}-1} \Bigg\langle G_0(Z,Z^\dagger)\ \mathrm{Tr}\,\left(\left(\prod_{u=1}^{r-1} Z^{k_{0,u}} {Z^\dagger}^{l_{0,u}}\right) Z^j\right) \\
& \qquad \mathrm{Tr}\,\left(Z^{k_{0,r}-j-1} {Z^\dagger}^{l_{0,r}} \left(\prod_{u=r+1}^{R_0} Z^{k_{0,u}} {Z^\dagger}^{l_{0,u}}\right)\right)\Bigg\rangle_C \\
& + \sum_{t=1}^{p}\sum_{r=1}^{R_t}\sum_{j=0}^{k_{t,r}-1} \Bigg\langle G_t(Z,Z^\dagger)\ \mathrm{Tr}\,\left(\left(\prod_{u=1}^{r-1} Z^{k_{t,u}} {Z^\dagger}^{l_{t,u}}\right) Z^j f_0(Z,Z^\dagger) Z^{k_{t,r}-j-1} {Z^\dagger}^{l_{t,r}}\right. \\
& \qquad \left.\left(\prod_{u=r+1}^{R_t} Z^{k_{t,u}} {Z^\dagger}^{l_{t,u}}\right)\right)\Bigg\rangle_C
\end{aligned} \tag{3-26}$$

and

$$\begin{aligned}
&\alpha_2 \left\langle G_0(Z,Z^\dagger)\,\mathrm{Tr}\, Z^\dagger f_0(Z,Z^\dagger)\right\rangle_C + \gamma \left\langle G_0(Z,Z^\dagger)\,\mathrm{Tr}\, Z f_0(Z,Z^\dagger))\right\rangle_C \\
= \quad & \sum_{r=1}^{R_0}\sum_{j=0}^{l_{0,r}-1} \Bigg\langle G_0(Z,Z^\dagger)\ \mathrm{Tr}\,\left(\left(\prod_{u=1}^{r-1} Z^{k_{0,u}} {Z^\dagger}^{l_{0,u}}\right) Z^{k_{0,r}} {Z^\dagger}^j\right) \\
& \qquad \mathrm{Tr}\,\left({Z^\dagger}^{l_{0,r}-j-1} \left(\prod_{u=r+1}^{R_0} Z^{k_{0,u}} {Z^\dagger}^{l_{0,u}}\right)\right)\Bigg\rangle_C \\
& + \sum_{t=1}^{p}\sum_{r=1}^{R_t}\sum_{j=0}^{l_{t,r}-1} \Bigg\langle G_t(Z,Z^\dagger)\ \mathrm{Tr}\,\left(\left(\prod_{u=1}^{r-1} Z^{k_{t,u}} {Z^\dagger}^{l_{t,u}}\right) Z^{k_{t,r}} {Z^\dagger}^j f_0(Z,Z^\dagger) {Z^\dagger}^{l_{t,r}-j-1}\right. \\
& \qquad \left.\left(\prod_{u=r+1}^{R_t} Z^{k_{t,u}} {Z^\dagger}^{l_{t,u}}\right)\right)\Bigg\rangle_C
\end{aligned} \tag{3-27}$$

Notice that the RHS is a linear combination of invariant polynomial functions of degree strictly lower than the LHS.

proof:

The proof is very similar to that of theorem 3.1. Write that the integral of a total derivative is zero:

$$0 = \int dZ \frac{\partial}{\partial \mathrm{Re} Z_{ij}} \left(f_{r,s}(Z,Z^\dagger)\, \mathrm{e}^{-\mathrm{Tr}\,(\frac{\alpha_1}{2}Z^2+\frac{\alpha_2}{2}{Z^\dagger}^2+\gamma Z Z^\dagger)}\right) \tag{3-28}$$

and

$$0 = -i\int dZ \frac{\partial}{\partial \mathrm{Im} Z_{ij}} \left(f_{r,s}(Z,Z^\dagger)\, \mathrm{e}^{-\mathrm{Tr}\,(\frac{\alpha_1}{2}Z^2+\frac{\alpha_2}{2}{Z^\dagger}^2+\gamma Z Z^\dagger)}\right)$$

$$(3-29)$$

Taking the sum of both lines, one can proceed as if all the Z_{ij} and ${Z^\dagger}_{ij}$ were real indepedent variables, and from there, follow the proof of theorem 3.1. $\square$

Remark 3.3 We see that the loop equations of both models are identical. It is clear from the above derivation that this is general, even for non gaussian measures. When the measure is gaussian, the loop equations determine completely every expectation value, while for non-gaussian measures, the loop equations give recursion relations for expectation values, but don't give the initial conditions.

Let us consider in particular the "semi-classical case" [4, 7], i.e. with a measure of the type

$$\partial\mu(M_1, M_2) = \mathrm{e}^{-\operatorname{Tr}[V_1(M_1)+V_2(M_2)+M_1M_2]} \tag{3-30}$$

where V_1' and V_2' are rational functions. In that case, the initial conditions which allow to determine all polynomial expectation values recursively, are in one–to–one correspondance with homology classes of integration paths for pairs of eigenvalues [7], therefore, there exists a choice of integration path Γ such that one can write:

$$\int_{(H_n\times H_n)(\Gamma)} dM_1\, dM_2\, \mathrm{e}^{-\operatorname{Tr}[V_1(M_1)+V_2(M_2)+M_1M_2]} \equiv \int_{GL_n(\mathrm{C})} dZ\, \mathrm{e}^{-\operatorname{Tr}[V_1(Z)+V_2(Z^\dagger)+ZZ^\dagger]} \tag{3-31}$$

and one can consider that this equality defines the RHS. Somehow, the complex matrix model is nothing but the analytical continuation of the 2-matrix model defined on some classes of contours.

3.3 Relation between the two models

Theorem 3.3 *For any polynomial invariant function $F(A,B)$, one has:*

$$\langle F\rangle_H = \langle F\rangle_C \tag{3-32}$$

Notice that $\langle F\rangle_H$ and $\langle F\rangle_C$ have been defined for different range of values of α_1, α_2, and γ, but, as we have explained above, both are polynomials of $\frac{\alpha_1}{\delta}$, $\frac{\alpha_2}{\delta}$ and $\frac{\gamma}{\delta}$ (and can be analyticaly continued to any α_1, α_2, and γ). Theorem 3.3 is thus an equality between polynomials.

proof:

It is sufficient to prove it for monomial invariant functions. The proof is clearly obtained from the loop equations, by recursion on $\deg F$. It is obviously true for $\deg F = 0$, i.e. $F = 1$. And the loop equations of both models are identical. $\square$

Definition 3.5 *For any two given complex diagonal matrices X and Y, and any polynomial invariant function F, define:*

$$\tilde{W}_F(X,Y) := \Delta^2(X)\,\Delta^2(Y) \int_{U(n)} dU\ F(X, UYU^\dagger)\,\mathrm{e}^{-\gamma \operatorname{Tr} XUYU^\dagger} \tag{3-33}$$

$$\omega_F(X,Y) := \Delta(X)\,\Delta(Y)\, \frac{\int_{T(n)} dT\ F(X+T, Y+T^\dagger)\,\mathrm{e}^{-\gamma\operatorname{Tr} TT^\dagger}}{\int_{T(n)} dT\,\mathrm{e}^{-\gamma \operatorname{Tr} TT^\dagger}} \tag{3-34}$$

which is a polynomial in all its variables X_i, Y_j, and a polynomial in $1/\gamma$, and:

$$W_F(X,Y) := \frac{1}{n!^2}\sum_\sigma\sum_\tau \Delta(X_\sigma)\Delta(Y_\tau)\,\mathrm{e}^{-\gamma\,\mathrm{Tr}\,X_\sigma Y_\tau}\int_{T(n)} dT\ F(X_\sigma+T, Y_\tau+T^\dagger)\,\mathrm{e}^{-\gamma\,\mathrm{Tr}\,TT^\dagger} \tag{3-35}$$

Theorem 3.4 *For any polynomial invariant function $F(A,B)$, one has:*

$$\begin{aligned} &\frac{\tilde{J}_n^2}{n!^2\,Z_H(n,\gamma,\alpha_1,\alpha_2)}\int_{D_n(\mathrm{R})\times D_n(R)} dXdY\ \mathrm{e}^{-\frac{\alpha_1}{2}\,\mathrm{Tr}\,X^2}\mathrm{e}^{-\frac{\alpha_2}{2}\,\mathrm{Tr}\,Y^2}\tilde{W}_F(X,Y) \\ =\ &\frac{J_n}{n!\,Z_C(n,\gamma,\alpha_1,\alpha_2)}\int_{D_n(\mathrm{C})} dX\ \mathrm{e}^{-\frac{\alpha_1}{2}\,\mathrm{Tr}\,X^2}\mathrm{e}^{-\frac{\alpha_2}{2}\,\mathrm{Tr}\,\overline{X}^2}W_F(X,\overline{X}) \end{aligned} \tag{3-36}$$

proof:

Start from theorem 3.3, diagonalize M_1 and M_2 on the hermitean side, and jordanize Z on the complex side.

$$\begin{aligned} \langle F\rangle_H &= \frac{\tilde{J}_n^2}{n!^2\,Z_H(n,\gamma,\alpha_1,\alpha_2)}\int_{D_n(\mathrm{R})\times D_n(R)} dXdY\ \mathrm{e}^{-\frac{\alpha_1}{2}\,\mathrm{Tr}\,X^2}\mathrm{e}^{-\frac{\alpha_2}{2}\,\mathrm{Tr}\,Y^2}\tilde{W}_F(X,Y) \\ = \langle F\rangle_C &= \frac{J_n}{n!\,Z_C(n,\gamma,\alpha_1,\alpha_2)}\int_{D_n(\mathrm{C})} dX\ \mathrm{e}^{-\frac{\alpha_1}{2}\,\mathrm{Tr}\,X^2}\mathrm{e}^{-\frac{\alpha_2}{2}\,\mathrm{Tr}\,\overline{X}^2}\mathrm{e}^{-\gamma\,\mathrm{Tr}\,X\overline{X}}\,\omega_F(X,\overline{X}) \\ &= \frac{J_n}{n!\,Z_C(n,\gamma,\alpha_1,\alpha_2)}\int_{D_n(\mathrm{C})} dX\ \mathrm{e}^{-\frac{\alpha_1}{2}\,\mathrm{Tr}\,X^2}\mathrm{e}^{-\frac{\alpha_2}{2}\,\mathrm{Tr}\,\overline{X}^2}\mathrm{e}^{-\gamma\,\mathrm{Tr}\,X_\sigma\overline{X}_\tau}\,\omega_F(X_\sigma,\overline{X}_\tau) \\ &= \frac{J_n}{n!\,Z_C(n,\gamma,\alpha_1,\alpha_2)}\int_{D_n(\mathrm{C})} dX\ \mathrm{e}^{-\frac{\alpha_1}{2}\,\mathrm{Tr}\,X^2}\mathrm{e}^{-\frac{\alpha_2}{2}\,\mathrm{Tr}\,\overline{X}^2}\,W_F(X_\sigma,\overline{X}_\tau) \end{aligned} \tag{3-37}$$

The equality in the first line is obtained by diagonalizing M_1 and M_2 (with Jacobian given in eq.2-17), the equality in the second line is obtained by Jordanizing Z (with Jacobian given in eq.2-19), the equality between the second and third line holds for any pair of permutations σ and τ (it can be proven with the Lemma A.1 given in appendix), and the equality of the last line comes from the definition of W_F. $\square$

4 Unitary group integrals

Here is one of the most important theorems of this paper:

4.1 Unitary integrals and triangular integrals

Theorem 4.1 *For any invariant function $F(A,B)$ one has:*

$$\begin{aligned} &\int_{U(n)} dU\,F(X,UYU^\dagger)\,\mathrm{e}^{-\gamma\,\mathrm{Tr}\,XUYU^\dagger} \\ =\ &\frac{c_n}{n!}\,\frac{\sum_\sigma\sum_\tau(-1)^\sigma(-1)^\tau\mathrm{e}^{-\gamma\,\mathrm{Tr}\,X_\sigma Y_\tau}\int_{T(n)}F(X_\sigma+T,Y_\tau+T^\dagger)\,\mathrm{e}^{-\gamma\,\mathrm{Tr}\,TT^\dagger}\,dT}{\Delta(X)\Delta(Y)} \end{aligned}$$

$(4-1)$

where

$$c_n = \frac{\prod_{k=0}^{n-1} k!}{(-2\pi)^{\frac{n(n-1)}{2}}} \tag{4-2}$$

i.e.

$$\tilde{W}_F(X,Y) = n!\, c_n\, W_F(X,Y) \tag{4-3}$$

proof:

Using the Lemma A.1 given in appendix, and using theorem 3.4, we have:

$$\begin{aligned}
& \int_{D_n(\mathrm{R})\times D_n(\mathrm{R})} dX\, dY\ \ \mathrm{e}^{-\frac{\alpha_1}{2}\operatorname{Tr} X^2} \mathrm{e}^{-\frac{\alpha_2}{2}\operatorname{Tr} Y^2} \tilde{W}_F(X,Y) \\
= \quad & \frac{n!^2\, Z_H}{\tilde{J}_n^2}\, \frac{J_n}{n!\, Z_C} \int_{D_n(\mathrm{C})} dX\ \ \mathrm{e}^{-\frac{\alpha_1}{2}\operatorname{Tr} X^2} \mathrm{e}^{-\frac{\alpha_2}{2}\operatorname{Tr} \overline{X}^2} W_F(X,\overline{X}) \\
= \quad & \frac{n!^2\, Z_H}{\tilde{J}_n^2}\, \frac{J_n}{n!\, Z_C}\, \frac{1}{n!^2} \sum_{\sigma,\tau} \int_{D_n(\mathrm{C})} dX\ \ \mathrm{e}^{-\frac{\alpha_1}{2}\operatorname{Tr} X^2} \mathrm{e}^{-\frac{\alpha_2}{2}\operatorname{Tr} \overline{X}^2} \mathrm{e}^{-\gamma \operatorname{Tr} X_\sigma \overline{X}_\tau} \omega_F(X_\sigma, \overline{X}_\tau) \\
= \quad & \frac{n!^2\, Z_H}{\tilde{J}_n^2 \left(\frac{2\pi}{\sqrt{\delta}}\right)^n}\, \frac{J_n \left(\frac{\pi}{\sqrt{-\delta}}\right)^n}{n!\, Z_C}\, \frac{1}{n!^2} \sum_{\sigma,\tau} \int_{D_n(\mathrm{R})\times D_n(\mathrm{R})} dX\, dY\ \ \mathrm{e}^{-\frac{\alpha_1}{2}\operatorname{Tr} X^2} \mathrm{e}^{-\frac{\alpha_2}{2}\operatorname{Tr} Y^2} \mathrm{e}^{-\gamma \operatorname{Tr} X_\sigma Y_\tau} \omega_F(X_\sigma, Y_\tau) \\
= \quad & \frac{n!^2\, Z_H}{\tilde{J}_n^2 \left(\frac{2\pi}{\sqrt{\delta}}\right)^n}\, \frac{J_n \left(\frac{\pi}{\sqrt{-\delta}}\right)^n}{n!\, Z_C} \int_{D_n(\mathrm{R})\times D_n(\mathrm{R})} dX\, dY\ \ \mathrm{e}^{-\frac{\alpha_1}{2}\operatorname{Tr} X^2} \mathrm{e}^{-\frac{\alpha_2}{2}\operatorname{Tr} Y^2} W_F(X,Y) \\
= \quad & n! c_n \int_{D_n(\mathrm{R})\times D_n(\mathrm{R})} dX\, dY\ \ \mathrm{e}^{-\frac{\alpha_1}{2}\operatorname{Tr} X^2} \mathrm{e}^{-\frac{\alpha_2}{2}\operatorname{Tr} Y^2} W_F(X,Y)
\end{aligned}$$

$(4-4)$

Notice that if $f(A)$ and $g(B)$ are invariant functions i.e. $f(UAU^{-1}) = f(A)$ for all A and U (resp. $g(UBU^{-1}) = g(B)$ for all B and U), one has:

$$W_{f(X)g(Y)F(X,Y)}(X,Y) = f(X)g(Y)W_F(X,Y)\ , \qquad \tilde{W}_{f(X)g(Y)F(X,Y)}(X,Y) = f(X)g(Y)\tilde{W}_F(X,Y) \tag{4-5}$$

Thus, for any f and g:

$$0 = \int_{D_n(\mathrm{R})\times D_n(\mathrm{R})} dX\, dY\ \mathrm{e}^{-\frac{\alpha_1}{2}\operatorname{Tr} X^2} \mathrm{e}^{-\frac{\alpha_2}{2}\operatorname{Tr} Y^2} f(X)g(Y)(n!\, c_n\, W_F(X,Y) - \tilde{W}_F(X,Y)) \tag{4-6}$$

Since $\tilde{W}_F(X,Y)$ and $n!c_nW_F(X,Y)$ are symmetric functions and entire functions of all their variables, they must be identicaly equal to one another. $\square$

4.2 Examples

Let us illustrate theorem 4.1 on some simple examples and recover some classical results.

4.2.1 Harish-Chandra–Itzykson–Zuber's formula

We can use theorem 4.1, to find a new proof of the famous Harish-Chandra–Itzykson–Zuber's formula [18, 20].

Indeed, consider $F(A,B)=1$, theorem 4.1 gives:

$$\begin{aligned}\int_{U(n)} \mathrm{e}^{-\gamma \operatorname{Tr} XUYU^\dagger} &= \frac{c_n}{n!}\,\frac{\sum_{\sigma,\tau}(-1)^\sigma\,(-1)^\tau\,\prod_i \mathrm{e}^{-\gamma X_{\sigma_i}Y_{\tau_i}}}{\Delta(X)\Delta(Y)} \int_{T_n} dT\,\mathrm{e}^{-\gamma \operatorname{Tr} TT^\dagger} \\ &= c_n \left(\frac{\pi}{\gamma}\right)^{\frac{n(n-1)}{2}} \frac{\det E}{\Delta(X)\Delta(Y)}\end{aligned} \tag{4-7}$$

which is the famous Harish Chandra-Itzykzon-Zuber integral. Here E is the matrix

$$E_{ij} := \mathrm{e}^{-\gamma X_i Y_j} \tag{4-8}$$

4.2.2 Morozov's formula

Consider $\operatorname{Tr} A^k B^l$ for any integers k and l. It is in fact simpler to introduce a generating function:

$$F(A,B) = \operatorname{Tr} \frac{1}{x-A}\frac{1}{y-B} = \sum_{k=0}^\infty\sum_{l=0}^\infty \frac{1}{x^{k+1}}\,\frac{1}{y^{l+1}}\,\operatorname{Tr} A^k B^l \tag{4-9}$$

which is to be understood as a formal power series in its large x and large y expansion. $F(A,B)$ is merely a convenient way of considering all polynomial invariant functions of type $\operatorname{Tr} A^k B^l$ at once.

We have:

$$\frac{1}{x-(X+T)} = \sum_{p=0}^n \left(\frac{1}{x-X}\,T\right)^p \frac{1}{x-X} \tag{4-10}$$

and thus:

$$\begin{aligned}&\frac{1}{\int_{T(n)} dT\,\mathrm{e}^{-\gamma \operatorname{Tr} TT^\dagger}} \int_{T(n)} \operatorname{Tr} \frac{1}{x-(X+T)}\,\frac{1}{y-(Y+T^\dagger)}\,dT\,\mathrm{e}^{-\gamma \operatorname{Tr} TT^\dagger} \\ =\;& \frac{1}{\int_{T(n)} dT\,\mathrm{e}^{-\gamma \operatorname{Tr} TT^\dagger}} \sum_{p=0}^n\sum_{q=0}^n \int_{T(n)} \operatorname{Tr} \left(\frac{1}{x-X}\,T\right)^p \frac{1}{x-X}\,\frac{1}{y-Y}\left(T^\dagger \frac{1}{y-Y}\right)^q dT\,\mathrm{e}^{-\gamma \operatorname{Tr} TT^\dagger} \\ =\;& \sum_{p=0}^n\sum_{q=0}^n \sum_{i_1<i_2\ldots<i_{p+1}}\;\sum_{j_1<j_2\ldots<j_{q+1}} \delta_{i_1,j_1}\delta_{i_p,j_q} \prod_{k=1}^{p+1}\frac{1}{x-X_{i_k}}\prod_{l=1}^{q+1}\frac{1}{y-Y_{j_l}} \\ &\frac{\int_{T(n)} T_{i_1,i_2}T_{i_2,i_3}\ldots T_{i_p,i_{p+1}}T^\dagger_{j_{q+1},j_q}\ldots T^\dagger_{j_2,j_1}\,dT\,\mathrm{e}^{-\gamma \operatorname{Tr} TT^\dagger}}{\int_{T(n)} dT\,\mathrm{e}^{-\gamma \operatorname{Tr} TT^\dagger}}\end{aligned} \tag{4-11}$$

That last integral is non vanishing only if $p=q$, and according to Wick's theorem, it is the sum of all possible pairings. Because of the ordering of the i_k's and j_l's, the only non vanishing pairing

is obtained for $i_k = j_k$ for all k. Therefore:

$$\begin{aligned}
& \frac{1}{\int_{T(n)} dT\, \mathrm{e}^{-\gamma \operatorname{Tr} TT^\dagger}} \int_{T(n)} \operatorname{Tr} \frac{1}{x-(X+T)} \frac{1}{y-(Y+T^\dagger)}\, dT\, \mathrm{e}^{-\gamma \operatorname{Tr} TT^\dagger} \\
&= \sum_{p=0}^{\infty} \sum_{i_1<i_2\ldots<i_{p+1}} \frac{1}{\gamma^p} \prod_{k=1}^{p+1} \frac{1}{(x-X_{i_k})(y-Y_{i_k})} \\
&= -\gamma + \gamma \prod_{i=1}^{n} \left(1 + \frac{1}{\gamma(x-X_i)(y-Y_i)}\right) \qquad \text{(4-12)}
\end{aligned}$$

and then theorem 4.1 gives:

$$\begin{aligned}
& \left(\frac{\gamma}{\pi}\right)^{\frac{n(n-1)}{2}} \int_{U(n)} dU\, \operatorname{Tr} \left(\frac{1}{x-X} U \frac{1}{y-Y} U^\dagger\right) \mathrm{e}^{-\gamma \operatorname{Tr} XUYU^\dagger} \\
&= \frac{c_n\, \gamma}{n!} \frac{\sum_{\sigma,\tau} (-1)^\sigma (-1)^\tau \left(-\prod_i \mathrm{e}^{-\gamma X_{\sigma_i} Y_{\tau_i}} + \prod_i \left(\mathrm{e}^{-\gamma X_{\sigma_i} Y_{\tau_i}} + \frac{1}{\gamma} \frac{1}{x-X_{\sigma_i}} \mathrm{e}^{-\gamma X_{\sigma_i} Y_{\tau_i}} \frac{1}{y-Y_{\tau_i}}\right)\right)}{\Delta(X)\Delta(Y)} \\
&= \gamma\, c_n \frac{-\det E + \det\left(E + \frac{1}{\gamma} \frac{1}{x-X} E \frac{1}{y-Y}\right)}{\Delta(X)\Delta(Y)} \\
&= \gamma \left(-1 + \det\left(1 + \frac{1}{\gamma} \frac{1}{x-X} E \frac{1}{y-Y} E^{-1}\right)\right) c_n \frac{\det E}{\Delta(X)\Delta(Y)} \qquad \text{(4-13)}
\end{aligned}$$

i.e.

$$\frac{\int_{U(n)} dU\, \operatorname{Tr}\left(\frac{1}{x-X} U \frac{1}{y-Y} U^\dagger\right) \mathrm{e}^{-\gamma \operatorname{Tr} XUYU^\dagger}}{\int_{U(n)} dU\, \mathrm{e}^{-\gamma \operatorname{Tr} XUYU^\dagger}} = \gamma \left(-1 + \det\left(1 + \frac{1}{\gamma} \frac{1}{x-X} E \frac{1}{y-Y} E^{-1}\right)\right) \qquad \text{(4-14)}$$

which is identical (for $\gamma = -1$) to what was found in [6, 13], i.e. the compact version of Morozov's formula [23].

5 Computation of triangular integrals

The goal of this section is to compute the triangular integral on the RHS of theorem 4.1. Here, we consider $\gamma = 1$.

5.1 Parametrization of polynomial invariant functions

Definition 5.1 *Let R be a positive integer. Let $\vec{x} = (x_1, \ldots, x_R)$ and $\vec{y} = (y_1, \ldots, y_R)$ be $2R$ complex numbers. Let π and π' be two permutations of Σ_R.*

The permutation $\pi\pi'^{-1}$ is made of p cycles $C_1, \ldots, C_p$ of lenght $R_1, \ldots, R_p$ which we note:

$$C_k = (i_{k,1} \xrightarrow{\pi} j_{k,1} \overset{\pi'^{-1}}{\rightsquigarrow} i_{k,2} \xrightarrow{\pi} j_{k,2} \overset{\pi'^{-1}}{\rightsquigarrow} i_{k,3} \xrightarrow{\pi} \ldots \overset{\pi'^{-1}}{\rightsquigarrow} i_{k,R_k} \xrightarrow{\pi} j_{k,R_k} \overset{\pi'^{-1}}{\rightsquigarrow} i_{k,1}) \qquad \text{(5-1)}$$

We define, for $(A,B)\in GL_n(\mathrm{C})^2$ in any dimension n:

$$F_{\pi,\pi'}(\vec{x},\vec{y},A,B) := \prod_{k=1}^{p}\left(\delta_{R_k,1} + \operatorname{Tr}\prod_{l=1}^{R_k}\frac{1}{x_{i_{k,l}}-A}\frac{1}{y_{j_{k,l}}-B}\right) \tag{5-2}$$

As explained above, this definition is to be understood as a formal power series in the large x_i and y_j expansions, it is merely a way of considering all polynomial invariant functions at once.

Examples: with $R=2$, we have:

$$\begin{aligned}
F_{(1)(2),(1)(2)}(x_1,x_2,y_1,y_2,A,B) &= \left(1+\operatorname{Tr}\frac{1}{x_1-A}\frac{1}{y_1-B}\right)\left(1+\operatorname{Tr}\frac{1}{x_2-A}\frac{1}{y_2-B}\right)\\
F_{(12),(12)}(x_1,x_2,y_1,y_2,A,B) &= \left(1+\operatorname{Tr}\frac{1}{x_1-A}\frac{1}{y_2-B}\right)\left(1+\operatorname{Tr}\frac{1}{x_2-A}\frac{1}{y_1-B}\right)\\
F_{(1)(2),(12)}(x_1,x_2,y_1,y_2,A,B) &= \operatorname{Tr}\frac{1}{x_1-A}\frac{1}{y_1-B}\frac{1}{x_2-A}\frac{1}{y_2-B}\\
F_{(12),(1)(2)}(x_1,x_2,y_1,y_2,A,B) &= \operatorname{Tr}\frac{1}{x_1-A}\frac{1}{y_2-B}\frac{1}{x_2-A}\frac{1}{y_1-B}
\end{aligned}$$

$(5-3)$

Definition 5.2 *Let R be a positive integer, $\vec{x}=(x_1,\dots,x_R)$ and $\vec{y}=(y_1,\dots,y_R)$ be $2R$ complex numbers. Let π and π' be two permutations of Σ_R. Let n be an integer, and $X=\operatorname{diag}(X_1,\dots,X_n)$ and $Y=\operatorname{diag}(Y_1,\dots,Y_n)$ be two complex diagonal matrices of size n, We define:*

$$W^{(n)}_{\pi,\pi'}(\vec{x},\vec{y},X,Y) := 1 \qquad \text{if } n=0 \text{ or } R=0 \tag{5-4}$$

$$W^{(n)}_{\pi,\pi'}(\vec{x},\vec{y},X,Y) := F_{\pi,\pi'}(\vec{x},\vec{y},X_1,Y_1) \qquad \text{if } n=1 \tag{5-5}$$

and otherwise

$$W^{(n)}_{\pi,\pi'}(\vec{x},\vec{y},X,Y) := \frac{\int_{T(n)} dT\, \mathrm{e}^{-\operatorname{Tr} TT^\dagger}\, F_{\pi,\pi'}(\vec{x},\vec{y},X+T,Y+T^\dagger)}{\int_{T(n)} dT\, \mathrm{e}^{-\operatorname{Tr} TT^\dagger}} \tag{5-6}$$

Here, $\frac{1}{x-(X+T)}$ is defined by:

$$\left(\frac{1}{x-(X+T)}\right)_{i,j} := \frac{\delta_{ij}}{x-X_i} + \sum_{p=1}^{(j-i)}\sum_{i<i_1<\dots<i_p<j}\frac{1}{x-X_i}T_{i,i_1}\frac{1}{x-X_{i_1}}T_{i_1,i_2}\dots\frac{1}{x-X_{i_p}}T_{i_p,j}\frac{1}{x-X_j} \tag{5-7}$$

5.2 Computation of triangular integrals of invariant functions

We are now going to find some recursion relation in n for the W's.

Theorem 5.1

$$W^{(n)}_{\pi,\pi'}(\vec{x},\vec{y},X,Y) = \sum_{\rho} \mathcal{M}^{(R)}_{\pi,\rho}(\vec{x},\vec{y},X_n,Y_n)\; W^{(n-1)}_{\rho,\pi'}(\vec{x},\vec{y},\tilde{X},\tilde{Y}) \tag{5-8}$$

where $\tilde{X} := \mathrm{diag}(X_1,\dots,X_{n-1})$, $\tilde{Y} := \mathrm{diag}(Y_1,\dots,Y_{n-1})$, *and:*

$$\mathcal{M}^{(R)}_{\pi,\rho}(\vec{x},\vec{y},X_n,Y_n) = \prod_{i=1}^{R}\left(\delta_{\pi(i),\rho(i)} + \frac{1}{(x_i-X_n)(y_{\pi(i)}-Y_n)}\right) \tag{5-9}$$

proof:

If T is a strictly upper triangular matrix of size n, we define $\tilde{T}$ the triangular matrix of size $n-1$, such that $\tilde{T}_{i,j} = T_{i,j}$ for all $i,j<n$, and $\vec{u}$ the vector made of the last column of T, $u_k = T_{k,n}$:

$$T = \begin{pmatrix} \ddots & \dots & \dots & \vdots & u_1 \\ & \ddots & \tilde{T} & \vdots & \vdots \\ & & \ddots & \vdots & \vdots \\ & & & \ddots & u_{n-1} \\ & & & & 0 \end{pmatrix} \tag{5-10}$$

We define $\left(\frac{1}{x-(\tilde{X}+\tilde{T})}\right)_{i,j} := 0$ if $i=n$ or $j=n$.

Notice that:

$$\left(\frac{1}{x-(X+T)}\right)_{i,j} = \left(\frac{1}{x-(\tilde{X}+\tilde{T})}\right)_{i,j} + \frac{\delta_{j,n}}{x-X_n}\sum_{k=1}^{n-1}\left(\frac{1}{x-(\tilde{X}+\tilde{T})}\right)_{i,k} u_k + \frac{\delta_{i,n}\delta_{j,n}}{x-X_n} \tag{5-11}$$

and

$$\begin{aligned} & 1+\ \mathrm{Tr}\,\frac{1}{x-(X+T)}\frac{1}{y-(Y+T^\dagger)} \\ = \quad & 1+\ \mathrm{Tr}\,\frac{1}{x-(\tilde{X}+\tilde{T})}\frac{1}{y-(\tilde{Y}+\tilde{T}^\dagger)} \\ & +\frac{1}{(x-X_n)(y-Y_n)}\left(1+\sum_{k=1}^{n-1}\sum_{l=1}^{n-1}\left(\frac{1}{y-(\tilde{Y}+\tilde{T}^\dagger)}\frac{1}{x-(\tilde{X}+\tilde{T})}\right)_{l,k} u_k\overline{u}_l\right) \\ (5-12) & \end{aligned}$$

Now, we integrate u out, using Wick's theorem, i.e. take the sum over all possible pairings of a u and a $\overline{u}$. The pairing $(u_k,\overline{u}_l)$ gives a factor $\delta_{k,l}$.

Let us represent W as a bivalent graph G, whose edges are pairs $(x_i, y_{\pi(i)})$, and whose vertices are pairs $(y_{\pi'(i)}, x_i)$.

Relation eq.5-11 means that, for each edge $(x_i, y_{\pi(i)})$ of G, we can either:

- let the edge untouched (first term in eq.5-11), with weight 1,
- remove the edge (second term in eq.5-11), with weight $\frac{1}{(x_i-X_n)(y_{\pi(i)}-Y_n)}$,

- remove the vertex $(y_{\pi'(i)},x_i)$ (third term in eq.5-11), with weight $\frac{1}{(x_i-X_n)(y_{\pi'(i)}-Y_n)}$, which means that either the neighboring edges cannot stay untouched.

Then, we integrate u out, i.e. we take the sum over all possible pairings, i.e. we draw new edges between vertices (those not removed), so that the final graph is bivalent. For each pairing, we get a new graph G'. The sum over possible pairings, is thus the sum over bivalent graphs G', whose vertices form a subset of the vertices of G, i.e.

$$W_G^{(n)} = \sum_{G'} \mathcal{M}_{G,G'}\, W_{G'}^{(n-1)} \tag{5-13}$$

where the coefficient $\mathcal{M}_{G,G'}$ is computed as follows:

- $\mathcal{M}_{G,G'}$ receives a factor $1+\frac{1}{(x_i-X_n)(y_{\pi(i)}-Y_n)}$ for each edge $(x_i,y_{\pi(i)})$ of G which is unchanged, i.e. which is an edge of G'. (1 if it was not removed, and $\frac{1}{(x_i-X_n)(y_{\pi(i)}-Y_n)}$ if it was removed and drawn again).

- the weight of each edge $(x_i,y_{\pi(i)})$ of G, which is not an edge of G', is $\frac{1}{(x_i-X_n)(y_{\pi(i)}-Y_n)}$.

- the weight of removing a vertex is the same as the weight of creating a lenght 1 cycle at that vertex. In other words, if G' has less vertices than G, consider G'' obtained from G' by adding lenght 1 cycles at each missing vertex, one has $\mathcal{M}_{G,G'}=\mathcal{M}_{G,G''}$. The sum over G' can thus be written as a sum over G'', where G'' has as many vertices as G, and all cycles of lenght 1 come together with a 1 added.

- relation eq.5-12 ensures that the previous rules apply also when G has lenght 1 cycles.

To sumarize, we have:

$$W_G^{(n)} = \sum_{G''} \mathcal{M}_{G,G''}\, W_{G''}^{(n-1)} \tag{5-14}$$

where

$$\mathcal{M}_{G,G''} = \prod_{(x_i,y_{\pi(i)})\in G''} \left(1+\frac{1}{(x_i-X_n)(y_{\pi(i)}-Y_n)}\right) \prod_{(x_i,y_{\pi(i)})\notin G''} \frac{1}{(x_i-X_n)(y_{\pi(i)}-Y_n)} \tag{5-15}$$

when G and G'' are written in terms of pairs of permutations, it reduces to eq.5-9. $\square$

Remark 5.1 Notice that:

$$\mathcal{M}^{(R)}(\vec{x},\vec{y},X_n,Y_n) = \mathcal{M}^{(R)}(\vec{x},\vec{y},X_n,Y_n)^t \tag{5-16}$$

$$\mathcal{M}^{(R)}_{\pi,\pi'}(\vec{x},\vec{y},X_n,Y_n) = \mathcal{M}^{(R)}_{\pi^{-1},\pi'^{-1}}(\vec{y},\vec{x},Y_n,X_n) \tag{5-17}$$

$$\mathcal{M}^{(R)}_{\pi\rho,\pi'\rho}(\vec{x},\vec{y},X_n,Y_n) = \mathcal{M}^{(R)}_{\pi,\pi'}(\vec{x}_{\rho^{-1}},\vec{y},X_n,Y_n) \tag{5-18}$$

Theorem 5.2

$$W^{(n)}_{\pi,\pi'}(\vec{x},\vec{y},X,Y) = \left(\mathcal{M}^{(R)}(\vec{x},\vec{y},X_n,Y_n)\,\mathcal{M}^{(R)}(\vec{x},\vec{y},X_{n-1},Y_{n-1})\,\dots\mathcal{M}^{(R)}(\vec{x},\vec{y},X_1,Y_1)\right)_{\pi,\pi'} \tag{5-19}$$

proof:

For $n=1$, we have

$$W^{(1)}_{\pi,\pi'}(\vec{x},\vec{y},X_1,Y_1) = F_{\pi,\pi'}(\vec{x},\vec{y},X_1,Y_1) = \mathcal{M}^{(R)}_{\pi,\pi'}(\vec{x},\vec{y},X_1,Y_1) \tag{5-20}$$

The proof follows from recursion on n. $\square$

Theorem 5.3 *The matrices $\mathcal{M}^{(R)}(\vec{x},\vec{y},\xi,\eta)$ commute among themselves:*

$$\mathcal{M}^{(R)}(\vec{x},\vec{y},\xi,\eta)\mathcal{M}^{(R)}(\vec{x},\vec{y},\xi',\eta') = \mathcal{M}^{(R)}(\vec{x},\vec{y},\xi',\eta')\mathcal{M}^{(R)}(\vec{x},\vec{y},\xi,\eta) \tag{5-21}$$

proof:

Let $n=2$, $X=\operatorname{diag}(X_1,X_2)$ and $Y=\operatorname{diag}(Y_1,Y_2)$ be two diagonal matrices, and $\tilde{X} = \operatorname{diag}(X_2,X_1)$ and $\tilde{Y}=\operatorname{diag}(Y_2,Y_1)$. Let T be a 2×2 upper triangular matrix with non vanishing element T_{12}. Let U be the 2×2 matrix:

$$U = \begin{pmatrix} \overline{T}_{12} & Y_2-Y_1 \\ X_1-X_2 & T_{12} \end{pmatrix} \tag{5-22}$$

it satisfies:

$$U(X+T) = (\tilde{X}+\overline{T})U \ , \qquad U(Y+T^\dagger) = (\tilde{Y}+T^t)U \tag{5-23}$$

If U is invertible (which is true for almost every T), one has:

$$F_{\pi,\rho}(\vec{x},\vec{y},X+T,Y+T^\dagger) = F_{\pi,\rho}(\vec{x},\vec{y},\tilde{X}+\overline{T},\tilde{Y}+T^t) \tag{5-24}$$

for every T (except a zero measure subset). Since the Jacobian $\left|\frac{\partial\overline{T}}{\partial T}\right| = 1$, one has:

$$\frac{\int_{T(2)} dT\,\mathrm{e}^{-\operatorname{Tr}TT^\dagger}\,F_{\pi,\rho}(\vec{x},\vec{y},X+T,Y+T^\dagger)}{\int_{T(2)} dT\,\mathrm{e}^{-\operatorname{Tr}TT^\dagger}} = \frac{\int_{T(2)} d\tilde{T}\,\mathrm{e}^{-\operatorname{Tr}\tilde{T}\tilde{T}^\dagger}\,F_{\pi,\rho}(\vec{x},\vec{y},\tilde{X}+\overline{T},\tilde{Y}+T^t)}{\int_{T(2)} d\tilde{T}\,\mathrm{e}^{-\operatorname{Tr}\tilde{T}\tilde{T}^\dagger}} \tag{5-25}$$

Using theorem 5.2 for $n=2$, we have:

$$\mathcal{M}^{(R)}(\vec{x},\vec{y},X_1,Y_1)\mathcal{M}^{(R)}(\vec{x},\vec{y},X_2,Y_2) = \mathcal{M}^{(R)}(\vec{x},\vec{y},X_2,Y_2)\mathcal{M}^{(R)}(\vec{x},\vec{y},X_1,Y_1) \tag{5-26}$$

$\square$

Corollary 5.1 *therefore, there exists an orthogonal matrix $\mathcal{U}(\vec{x},\vec{y})$, independent of ξ and η, such that:*

$$\Lambda(\vec{x},\vec{y},\xi,\eta) := \mathcal{U}(\vec{x},\vec{y})\,\mathcal{M}^{(R)}(\vec{x},\vec{y},\xi,\eta)\,\mathcal{U}^t(\vec{x},\vec{y}) \tag{5-27}$$

is a diagonal matrix

$$\Lambda(\vec{x},\vec{y},\xi,\eta) = \operatorname{diag}\left(\Lambda_\pi(\vec{x},\vec{y},\xi,\eta)\right) \tag{5-28}$$

Notice that Λ is a rational function of ξ and η.

Thus:

$$\frac{\int_{T(n)} dT\, \mathrm{e}^{-\operatorname{Tr} TT^\dagger}\, F_{\pi,\pi'}(\vec{x},\vec{y},X+T,Y+T^\dagger)}{\int_{T(n)} dT\, \mathrm{e}^{-\operatorname{Tr} TT^\dagger}} = \sum_\rho \mathcal{U}_{\pi,\rho}(\vec{x},\vec{y})\,\mathcal{U}_{\pi',\rho}(\vec{x},\vec{y}) \prod_{i=1}^n \Lambda_\rho(\vec{x},\vec{y},X_i,Y_i) \quad (5\text{-}29)$$

and:

$$\frac{\int_{U(n)} dU\, \mathrm{e}^{-\operatorname{Tr} XUYU^\dagger}\, F_{\pi,\pi'}(\vec{x},\vec{y},X,UYU^\dagger)}{\int_{U(n)} dU\, \mathrm{e}^{-\operatorname{Tr} XUYU^\dagger}} = \sum_\rho \mathcal{U}_{\pi,\rho}(\vec{x},\vec{y})\,\mathcal{U}_{\pi',\rho}(\vec{x},\vec{y}) \frac{\det\left(\mathrm{e}^{-X_iY_j}\Lambda_\rho(\vec{x},\vec{y},X_i,Y_j)\right)}{\det\left(\mathrm{e}^{-X_iY_j}\right)} \quad (5\text{-}30)$$

Remark 5.2 if one defines the "Matricial determinant" as follows:

Definition 5.3 *Let $M \in GL_n(Gl_m(\mathrm{C}))$, i.e. for each $i=1,\dots,n$, $j=1,\dots,n$, $M_{i,j}$ is a square matrices of size m. We define:*

$$\mathrm{Mdet}(M) := \frac{1}{n!}\sum_{\sigma\in\Sigma(n)}\sum_{\tau\in\Sigma(n)} (-1)^\sigma(-1)^\tau \prod_{i=1}^n M_{\sigma(i),\tau(i)} \quad (5\text{-}31)$$

which is a $m\times m$ square matrix.

then we have:

$$\int_{U(n)} dU\, F_{\pi,\pi'}(\vec{x},\vec{y},X,UYU^\dagger)\,\mathrm{e}^{-\operatorname{Tr} XUYU^\dagger} = c_n\ (\pi)^{\frac{n(n-1)}{2}}\ \frac{\left(\mathrm{Mdet}\left(\mathrm{e}^{-X_iY_j}\,\mathcal{M}^{(R)}(\vec{x},\vec{y},X_i,Y_j)\right)\right)_{\pi,\pi'}}{\Delta(X)\Delta(Y)} \quad (5\text{-}32)$$

if $R=0$, one immediately recovers the Itzykson–Zuber's formula, and if $R=1$, one immediately recovers Morozov's formula.

5.3 Examples

- Example $R=1$:

$$\mathcal{M}^{(1)}_{\mathbf{1},\mathbf{1}}(x,y,\xi,\eta) = 1 + \frac{1}{x-\xi}\frac{1}{y-\eta} \quad (5\text{-}33)$$

and thus:

$$\frac{\int_{T(n)} dT\, \mathrm{e}^{-\operatorname{Tr} TT^\dagger}\ \left(1+\ \operatorname{Tr}\frac{1}{x-(X+T)}\frac{1}{x-(Y+T^\dagger)}\right)}{\int_{T(n)} dT\, \mathrm{e}^{-\operatorname{Tr} TT^\dagger}} = \prod_{i=1}^n\left(1+\frac{1}{x-X_i}\frac{1}{y-Y_i}\right) \quad (5\text{-}34)$$

- Example $R=2$:

We have:

$$F_{(1)(2),(1)(2)}(x_1,x_2,y_1,y_2,A,B) = \left(1+\ \operatorname{Tr}\frac{1}{x_1-A}\frac{1}{y_1-B}\right)\left(1+\ \operatorname{Tr}\frac{1}{x_2-A}\frac{1}{y_2-B}\right)$$
$$F_{(12),(12)}(x_1,x_2,y_1,y_2,A,B) = \left(1+\ \operatorname{Tr}\frac{1}{x_1-A}\frac{1}{y_2-B}\right)\left(1+\ \operatorname{Tr}\frac{1}{x_2-A}\frac{1}{y_1-B}\right)$$

$$F_{(1)(2),(12)}(x_1,x_2,y_1,y_2,A,B) = \mathrm{Tr}\, \frac{1}{x_1-A}\frac{1}{y_1-B}\frac{1}{x_2-A}\frac{1}{y_2-B}$$
$$F_{(12),(1)(2)}(x_1,x_2,y_1,y_2,A,B) = \mathrm{Tr}\, \frac{1}{x_1-A}\frac{1}{y_2-B}\frac{1}{x_2-A}\frac{1}{y_1-B}$$
$$(5-35)$$

and

$$\begin{cases} \mathcal{M}^{(2)}_{(1)(2),(1)(2)}(x_1,x_2,y_1,y_2,\xi,\eta) = \left(1+\frac{1}{x_1-\xi}\frac{1}{y_1-\eta}\right)\left(1+\frac{1}{x_2-\xi}\frac{1}{y_2-\eta}\right) \\ \mathcal{M}^{(2)}_{(12),(12)}(x_1,x_2,y_1,y_2,\xi,\eta) = \left(1+\frac{1}{x_1-\xi}\frac{1}{y_2-\eta}\right)\left(1+\frac{1}{x_2-\xi}\frac{1}{y_1-\eta}\right) \\ \mathcal{M}^{(2)}_{(1)(2),(12)}(x_1,x_2,y_1,y_2,\xi,\eta) = \frac{1}{x_1-\xi}\frac{1}{y_1-\eta}\frac{1}{x_2-\xi}\frac{1}{y_2-\eta} \\ \mathcal{M}^{(2)}_{(12),(1)(2)}(x_1,x_2,y_1,y_2,\xi,\eta) = \frac{1}{x_1-\xi}\frac{1}{y_2-\eta}\frac{1}{x_2-\xi}\frac{1}{y_1-\eta} \end{cases} \tag{5-36}$$

i.e., the matrix $\mathcal{M}^{(2)}(x_1,x_2,y_1,y_2,\xi,\eta)$ is:

$$\begin{pmatrix} \left(1+\frac{1}{x_1-\xi}\frac{1}{y_1-\eta}\right)\left(1+\frac{1}{x_2-\xi}\frac{1}{y_2-\eta}\right) & \frac{1}{x_1-\xi}\frac{1}{y_1-\eta}\frac{1}{x_2-\xi}\frac{1}{y_2-\eta} \\ \frac{1}{x_1-\xi}\frac{1}{y_2-\eta}\frac{1}{x_2-\xi}\frac{1}{y_1-\eta} & \left(1+\frac{1}{x_1-\xi}\frac{1}{y_2-\eta}\right)\left(1+\frac{1}{x_2-\xi}\frac{1}{y_1-\eta}\right) \end{pmatrix} \tag{5-37}$$

i.e.

$$\begin{aligned} \mathcal{M}^{(2)}(x_1,x_2,y_1,y_2,\xi,\eta) \quad = \quad & \left(1+\frac{1}{2}\left(\frac{1}{x_1-\xi}+\frac{1}{x_2-\xi}\right)\left(\frac{1}{y_1-\eta}+\frac{1}{y_2-\eta}\right)\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ & +\frac{1}{(x_1-\xi)(x_2-\xi)(y_1-\eta)(y_2-\eta)}\begin{pmatrix} 1+S & 1 \\ 1 & 1-S \end{pmatrix} \end{aligned}$$
$$(5-38)$$

where

$$S = \frac{1}{2}(x_1-x_2)(y_1-y_2) \tag{5-39}$$

Define the following orthogonal matrix ($\mathcal{U}^{(2)}(x_1,x_2,y_1,y_2)\,\mathcal{U}^{(2)}(x_1,x_2,y_1,y_2)^t = \mathbf{1}$):

$$\mathcal{U}^{(2)}(x_1,x_2,y_1,y_2) := \frac{1}{\sqrt{2\lambda(\lambda-S)}}\begin{pmatrix} 1 & \lambda-S \\ S-\lambda & 1 \end{pmatrix}\,, \qquad \text{where } \lambda=\sqrt{1+S^2} \tag{5-40}$$

one has:

$$\mathcal{M}^{(2)}(x_1,x_2,y_1,y_2,\xi,\eta) = \mathcal{U}^{(2)}(x_1,x_2,y_1,y_2)\,\Lambda^{(2)}(x_1,x_2,y_1,y_2,\xi,\eta)\,\mathcal{U}^{(2)}(x_1,x_2,y_1,y_2)^t \tag{5-41}$$

where $\Lambda^{(2)}(x_1,x_2,y_1,y_2,\xi,\eta) = \mathrm{diag}(\Lambda_+(x_1,x_2,y_1,y_2,\xi,\eta),\Lambda_-(x_1,x_2,y_1,y_2,\xi,\eta)\,)$ with

$$\begin{aligned} \Lambda_\pm(x_1,x_2,y_1,y_2,\xi,\eta) \quad = \quad & 1+\frac{1}{2}\left(\frac{1}{x_1-\xi}+\frac{1}{x_2-\xi}\right)\left(\frac{1}{y_1-\eta}+\frac{1}{y_2-\eta}\right) \\ & +\frac{1\pm\lambda}{(x_1-\xi)(x_2-\xi)(y_1-\eta)(y_2-\eta)} \end{aligned} \tag{5-42}$$

Eventualy, one gets:

$$\frac{\int_{T(n)} dT\, \mathrm{e}^{-\mathrm{Tr}\, TT^\dagger}\ \mathrm{Tr}\, \frac{1}{x_1-(X+T)}\frac{1}{y_1-(Y+T^\dagger)}\frac{1}{x_2-(X+T)}\frac{1}{y_2-(Y+T^\dagger)}}{\int_{T(n)} dT\, \mathrm{e}^{-\mathrm{Tr}\, TT^\dagger}}$$

$$= \frac{1}{2\lambda}\left(\prod_{i=1}^{n}\Lambda_-(x_1,x_2,y_1,y_2,X_i,Y_i) - \prod_{i=1}^{n}\Lambda_+(x_1,x_2,y_1,y_2,X_i,Y_i)\right) \tag{5-43}$$

$$\frac{\int_{T(n)} dT\, \mathrm{e}^{-\mathrm{Tr}\, TT^\dagger}\left(1+\mathrm{Tr}\,\frac{1}{x_1-(X+T)}\frac{1}{y_1-(Y+T^\dagger)}\right)\left(1+\mathrm{Tr}\,\frac{1}{x_2-(X+T)}\frac{1}{y_2-(Y+T^\dagger)}\right)}{\int_{T(n)} dT\, \mathrm{e}^{-\mathrm{Tr}\, TT^\dagger}}$$
$$= \frac{1}{2\lambda}\left((\lambda+S)\prod_{i=1}^{n}\Lambda_+(x_1,x_2,y_1,y_2,X_i,Y_i) + (\lambda-S)\prod_{i=1}^{n}\Lambda_-(x_1,x_2,y_1,y_2,X_i,Y_i)\right) \tag{5-44}$$

6 Mixed correlation functions and biorthogonal polynomials

Let us consider two polynomial potentials $V_1(x)$ and $V_2(y)$. Our goal is to compute the following matrix expectation values:

$$\frac{\int_{H_n\times H_n} dM_1\, dM_2\, F_{\pi,\pi'}(\vec{x},\vec{y},M_1,M_2)\, \mathrm{e}^{-\mathrm{Tr}\,(V_1(M_1)+V_2(M_2)+M_1M_2)}}{\int_{H_n\times H_n} dM_1\, dM_2\, \mathrm{e}^{-\mathrm{Tr}\,(V_1(M_1)+V_2(M_2)+M_1M_2)}} \tag{6-1}$$

6.1 Biorthonormal polynomials

We recall here a few elementary notions about biorthogonal polynomials. More detailed descriptions can be found in particular in [22, 21, 6, 9, 8, 5].

We introduce two families of polynomials $p_n(x) = \frac{1}{\sqrt{h_n}}x^n + O(x^{n-1})$, $q_n(y) = \frac{1}{\sqrt{h_n}}y^n + O(y^{n-1})$, with the same leading coefficient $\frac{1}{\sqrt{h_n}}$, and orthonormal with respect to the pairing:

$$(p_n, q_m) = \int\int dx\, dy\, p_n(x)\, q_m(y)\, \mathrm{e}^{-(V_1(x)+V_2(y)+xy)} = \delta_{nm} \tag{6-2}$$

The integration path is a priori R × R, but this condition can be relaxed (see [4, 8]). When they exist, the biorthonormal polynomials are uniquely determined.

Since the biorthonormal polynomials form a basis, one can decompose $xp_n(x)$ onto the basis of $p_m(x)$ with $m \leq n+1$:

$$x\, p_n(x) = \sum_{m=0}^{n+1} Q_{nm}\, p_m(x) \tag{6-3}$$

and similarly:

$$y\, q_n(y) = \sum_{m=0}^{n+1} P_{nm}\, q_m(y) \tag{6-4}$$

Q and P are infinite matrices. In the case where V_2 (resp. V_1) is a polynomial, then Q (resp. P) is a finite band matrix.

We also introduce the following $\infty \times n$ rectangular matrix:

$$\Pi_{n-1} := \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & & \\ & 0 & \\ & & \end{pmatrix} \tag{6-5}$$

which is the projector onto the n first polynomials.

6.2 Mixed correlation functions

Theorem 6.1

$$\begin{aligned} & \frac{\int_{H_n\times H_n} dM_1\, dM_2\, F_{\pi,\pi'}(\vec{x},\vec{y},M_1,M_2)\, \mathrm{e}^{-\operatorname{Tr}(V_1(M_1)+V_2(M_2)+M_1M_2)}}{\int_{H_n\times H_n} dM_1\, dM_2\, \mathrm{e}^{-\operatorname{Tr}(V_1(M_1)+V_2(M_2)+M_1M_2)}} \\ = & \sum_\rho \mathcal{U}_{\pi,\rho}(\vec{x},\vec{y})\,\mathcal{U}_{\pi',\rho}(\vec{x},\vec{y}) \det\left(\Pi^t_{n-1} : \Lambda_\rho(\vec{x},\vec{y},Q,P^t) : \Pi_{n-1}\right) \end{aligned} \tag{6-6}$$

where for any function of two variables $f(\xi,\eta)$, we define : $f(Q,P^t)$: by putting the Q's on the right of the P's. This is always possible in this case because $\Lambda_\rho(\vec{x},\vec{y},\xi,\eta)$ is a rational function of ξ and η.

proof:

It works as usual (see [21, 22]), by writings Vandermonde determinants as:

$$\Delta(X) = \det(X_i^{j-1}) = \det(\sqrt{h_{j-1}}\, p_{j-1}(X_i)) = \prod_{i=0}^{n-1}\sqrt{h_i}\,\sum_\sigma (-1)^\sigma \prod_i p_{\sigma(i)}(X_i) \tag{6-7}$$

$$\Delta(Y) = \det(Y_i^{j-1}) = \det(\sqrt{h_{j-1}}\, q_{j-1}(Y_i)) = \prod_{i=0}^{n-1}\sqrt{h_i}\,\sum_\tau (-1)^\tau \prod_i q_{\tau(i)}(Y_i) \tag{6-8}$$

Then, we use eq.5-30, i.e.

$$\begin{aligned} & \frac{1}{c_n\,\tilde{J}_n^2\,\pi^{\frac{n(n-1)}{2}}} \int_{H_n\times H_n} dM_1\, dM_2\, F_{\pi,\pi'}(\vec{x},\vec{y},M_1,M_2)\, \mathrm{e}^{-\operatorname{Tr}(V_1(M_1)+V_2(M_2)+M_1M_2)} \\ = & \frac{1}{n!^2}\prod_{i=0}^{n-1} h_i \sum_{\rho\in\Sigma(R)} \mathcal{U}_{\pi,\rho}(\vec{x},\vec{y})\,\mathcal{U}_{\pi',\rho}(\vec{x},\vec{y}) \sum_{\sigma,\tau,\nu\in\Sigma(n)} (-1)^{\sigma\tau\nu} \\ & \int \prod_i \Lambda_\rho(\vec{x},\vec{y},X_i,Y_{\nu(i)}) p_{\sigma(i)}(X_i)\mathrm{e}^{-V_1(X_i)} q_{\tau\nu(i)}(Y_{\nu(i)})\mathrm{e}^{-V_2(Y_{\nu(i)})}\mathrm{e}^{-X_iY_{\nu(i)}}\, dX_i dY_{\nu(i)} \\ = & \frac{1}{n!^2}\prod_{i=0}^{n-1} h_i \sum_{\rho\in\Sigma(R)} \mathcal{U}_{\pi,\rho}(\vec{x},\vec{y})\,\mathcal{U}_{\pi',\rho}(\vec{x},\vec{y}) \sum_{\sigma,\tau,\nu\in\Sigma(n)} (-1)^{\sigma\tau\nu} \prod_{i=1}^{n} : \Lambda_\rho(\vec{x},\vec{y},Q,P^t) :_{\sigma(i),\tau\nu(i)} \end{aligned}$$

$$\begin{aligned}
&= \prod_{i=0}^{n-1} h_i \sum_{\rho\in\Sigma(R)} \mathcal{U}_{\pi,\rho}(\vec{x},\vec{y})\mathcal{U}_{\pi',\rho}(\vec{x},\vec{y}) \sum_{\sigma\in\Sigma(n)} (-1)^\sigma \prod_{i=1}^{n} :\Lambda_\rho(\vec{x},\vec{y},Q,P^t):_{i,\sigma(i)} \\
&= \prod_{i=0}^{n-1} h_i \sum_{\rho\in\Sigma(R)} \mathcal{U}_{\pi,\rho}(\vec{x},\vec{y})\mathcal{U}_{\pi',\rho}(\vec{x},\vec{y}) \det\left(\Pi_{n-1}^t : \Lambda_\rho(\vec{x},\vec{y},Q,P^t) : \Pi_{n-1}\right)
\end{aligned}$$

(6 − 9)

□

or, using the matricial determinant defined in def.5.3:

$$\frac{\int_{H_n\times H_n} dM_1\, dM_2\, F_{\pi,\pi'}(\vec{x},\vec{y},M_1,M_2)\,\mathrm{e}^{-\mathrm{Tr}\,(V_1(M_1)+V_2(M_2)+M_1M_2)}}{\int_{H_n\times H_n} dM_1\, dM_2\,\mathrm{e}^{-\mathrm{Tr}\,(V_1(M_1)+V_2(M_2)+M_1M_2)}} = \left(\mathrm{Mdet}\left(\Pi_{n-1}^t : \mathcal{M}^{(R)}(\vec{x},\vec{y},Q,P^t) : \Pi_{n-1}\right)\right)_{\pi,\pi'} \tag{6-10}$$

Example: with $R = 1$, we find:

$$\frac{\int_{H_n\times H_n} dM_1\, dM_2\,(1+\ \mathrm{Tr}\,\frac{1}{x-M_1}\frac{1}{y-M_2})\,\mathrm{e}^{-\mathrm{Tr}\,(V_1(M_1)+V_2(M_2)+M_1M_2)}}{\int_{H_n\times H_n} dM_1\, dM_2\,\mathrm{e}^{-\mathrm{Tr}\,(V_1(M_1)+V_2(M_2)+M_1M_2)}} = \det\left(\Pi_{n-1}^t\left(1+\frac{1}{y-P^t}\frac{1}{x-Q}\right)\Pi_{n-1}\right) \tag{6-11}$$

which is identical to what was found in [6].

7 Conclusions

In this article, we have shown that the hermitean 2-matrix model and the complex matrix model have the same loop equations. In the gaussian case, that implies they are identical. In case the weight is non–gaussian, the loop equations, which are recursion equations, determine all correlation functions when some initial conditions (moduli) are fixed. The generalization of the hermitean 2-matrix model to homology classes of contours (as in [8]), allows to have any arbitrary initial condidtions, so, there exists a choice of homology class of contours for each set of initial conditions, i.e. for which the complex matrix model is identical to the 2-hermitean matrix model. Conversely, the initial conditions for the complex matrix model are not fully understood yet, they depend on how the complex matrix model is defined. If the complex matrix model is only a formal integral defined by its large n properties as in [27, 28], initial conditions are associated to filling fractions, and can thus be chosen arbitrarily. If the complex matrix model is defined as the result of a convergent integral for all n, it is not known yet how to find which homology class of contours it corresponds to.

The consequence of that identification, through diagonalization of hermitean matrices and Jordanization of complex matrices, yields an identity between unitary group integrals and triangular matrices integrals, which seems to be a special case of the identification of $GL_n(\mathrm{C})/T(n)$ and

the quotient of $SU(n)$ by its Cartan subalgebra. The nature of that identification needs to be further understood, in particular in terms of characters of both groups, and in terms of group representation theory, in terms of Weyl's character formula, or Harish-Chandra formulae.

The gaussian triangular matrix integrals are easily computed, and we thus get very explicit expressions for all expectation values of the type which were studied by Shatashvili [24]. In particular, we have provided a new proof of the Itzykson-Zuber-Harish-Chandra integral, as well as Morozov's integral. The key piece in this computation is that the matrices $\mathcal{M}$ commute together. This fact seems to be related to some Yang-Baxter relations, and it would be interesting to understand how.

It would be interesting also to understand these formulae in the framework of Duistermaat-Heckman semiclassical theories [11].

Then, we have been able to perform the integral over eigenvalues, in a way very similar to what was done in [6], i.e. in terms of $n \times n$ determinants. It would then be interesting to rewrite these $n \times n$ determinants in terms of determinants of size independent of n, using kernels, as it is known for non-mixed expectations values (see [2, 3, 16]).

Aknowledgements: The authors want to thank F. David, P. Di Francesco, J.B. Zuber for stimulating discussions. One of the authors (B.E.) wants to thank the european network Enigma (MRTN-CT-2004-5652). A. P-F. wants to thank the SPhT Saclay for its hospitality when part of this work was being conducted, and the support of CIRIT grant 2001FI-00387.

Appendix A Gaussian integrals

Let $\delta = \alpha_1\alpha_2 - \gamma^2$.

Real integrals:

$$\int_{\mathrm{R}\times\mathrm{R}} dx\, dy\, \mathrm{e}^{-(\frac{\alpha_1}{2}x^2+\frac{\alpha_2}{2}y^2+\gamma xy)} = \frac{2\pi}{\sqrt{\delta}} \tag{1-1}$$

$$\begin{aligned}
\frac{\int_{\mathrm{R}\times\mathrm{R}} dx\, dy\, x^k y^l\, \mathrm{e}^{-(\frac{\alpha_1}{2}x^2+\frac{\alpha_2}{2}y^2+\gamma xy)}}{\int_{\mathrm{R}\times\mathrm{R}} dx\, dy\, \mathrm{e}^{-(\frac{\alpha_1}{2}x^2+\frac{\alpha_2}{2}y^2+\gamma xy)}} &= 0 \qquad \text{if } k+l \text{ is odd} \\
&= \left(\frac{\sqrt{\delta}}{2\pi}\right)\left(-2\frac{\partial}{\partial\alpha_1}\right)^{\frac{k-l}{2}}\left(-\frac{\partial}{\partial\gamma}\right)^{l}\left(\frac{2\pi}{\sqrt{\delta}}\right) \qquad \text{if } k \geq l \\
&= \left(\frac{\sqrt{\delta}}{2\pi}\right)\left(-2\frac{\partial}{\partial\alpha_2}\right)^{\frac{l-k}{2}}\left(-\frac{\partial}{\partial\gamma}\right)^{k}\left(\frac{2\pi}{\sqrt{\delta}}\right) \qquad \text{if } k \leq l
\end{aligned}$$

$(1-2)$

$$\int_{D_n(\mathrm{R})\times D_n(\mathrm{R})} dX\, dY\, \mathrm{e}^{-\,\mathrm{Tr}\,(\frac{\alpha_1}{2}X^2+\frac{\alpha_2}{2}Y^2+\gamma XY)} = \left(\frac{2\pi}{\sqrt{\delta}}\right)^n \tag{1-3}$$

Complex integrals:

$$\int_{\mathrm{C}} dx\, \mathrm{e}^{-(\frac{\alpha_1}{2}x^2+\frac{\alpha_2}{2}\overline{x}^2+\gamma x\overline{x})} = \frac{\pi}{\sqrt{-\delta}} \tag{1-4}$$

$$\begin{aligned}
\frac{\int_{\mathrm{C}} dx\ x^k\,\overline{x}^l\,\mathrm{e}^{-(\frac{\alpha_1}{2}x^2+\frac{\alpha_2}{2}\overline{x}^2+\gamma x\overline{x})}}{\int_{\mathrm{C}} dx\,\mathrm{e}^{-(\frac{\alpha_1}{2}x^2+\frac{\alpha_2}{2}\overline{x}^2+\gamma x\overline{x})}} &= 0 \qquad \text{if } k+l \text{ is odd} \\
&= \left(\frac{\sqrt{-\delta}}{\pi}\right)\left(-2\frac{\partial}{\partial\alpha_1}\right)^{\frac{k-l}{2}}\left(-\frac{\partial}{\partial\gamma}\right)^{l}\left(\frac{\pi}{\sqrt{-\delta}}\right) \qquad \text{if } k\geq l \\
&= \left(\frac{\sqrt{-\delta}}{\pi}\right)\left(-2\frac{\partial}{\partial\alpha_2}\right)^{\frac{l-k}{2}}\left(-\frac{\partial}{\partial\gamma}\right)^{k}\left(\frac{\pi}{\sqrt{-\delta}}\right) \qquad \text{if } k\leq l \\
&(1-5)
\end{aligned}$$

$$\int_{D_n(\mathrm{C})} dX\,\mathrm{e}^{-\,\mathrm{Tr}\,(\frac{\alpha_1}{2}X^2+\frac{\alpha_2}{2}\overline{X}^2+\gamma X\overline{X})} = \left(\frac{\pi}{\sqrt{-\delta}}\right)^n \tag{1-6}$$

Lemma A.1 *Let $\omega(X,Y)$, be a polynomial in all its variables $X_1,\dots,X_n$ and $Y_1,\dots,Y_n$, one has:*

$$\frac{\int_{D_n(\mathrm{R})\times D_n(\mathrm{R})} dX\,dY\,\omega(X,Y)\,\mathrm{e}^{-\,\mathrm{Tr}\,(\frac{\alpha_1}{2}X^2+\frac{\alpha_2}{2}Y^2+\gamma XY)}}{\int_{D_n(\mathrm{R})\times D_n(\mathrm{R})} dX\,dY\,\mathrm{e}^{-\,\mathrm{Tr}\,(\frac{\alpha_1}{2}X^2+\frac{\alpha_2}{2}Y^2+\gamma XY)}} = \frac{\int_{D_n(\mathrm{C})} dX\,\omega(X,\overline{X})\ \mathrm{e}^{-\,\mathrm{Tr}\,(\frac{\alpha_1}{2}X^2+\frac{\alpha_2}{2}\overline{X}^2+\gamma X\overline{X})}}{\int_{D_n(\mathrm{C})} dX\ \mathrm{e}^{-\,\mathrm{Tr}\,(\frac{\alpha_1}{2}X^2+\frac{\alpha_2}{2}\overline{X}^2+\gamma X\overline{X})}} \tag{1-7}$$

proof:

Eqs 1-2 and 1-5 show that it is true for $n=1$. By decomposing ω into monomials, the integral decouples into a product of $n=1$ type integrals. $\square$

Appendix B Some Commutations

Theorem B.1 *The matrix $\mathcal{M}^{(R)}(\vec{x},\vec{y},\xi,\eta)$ commutes with the matrix $\mathcal{A}(\vec{x},\vec{y})$ defined by:*

$$\left\{\begin{array}{l}
\mathcal{A}_{\pi,\pi}(\vec{x},\vec{y}) := \sum_i x_i y_{\pi(i)} \\
\mathcal{A}_{\pi,\pi'}(\vec{x},\vec{y}) := 1 \quad \text{if } \pi\pi'^{-1} = \text{transposition} \\
\mathcal{A}_{\pi,\pi'}(\vec{x},\vec{y}) := 0 \quad \text{otherwise}
\end{array}\right. \tag{2-1}$$

Theorem B.2 *The matrices $\mathcal{A}^{\alpha,\beta}(\vec{x},\vec{y})$ defined by:*

$$\begin{aligned}
\mathcal{A}^{\alpha,\beta}_{\pi,\pi'}(\vec{x},\vec{y}) \quad := \quad & \delta_{\beta,\pi(\alpha)}\prod_{i\neq\alpha}\left(\delta_{\pi(i),\pi'(i)}+\frac{1}{x_\alpha-x_i}\frac{1}{y_\beta-y_{\pi(i)}}\right) \\
&+\frac{1-\delta_{\beta,\pi(\alpha)}}{(x_\alpha-x_{\pi^{-1}(\beta)})(y_\beta-y_{\pi(\alpha)})}\prod_{i\neq\alpha,\pi^{-1}(\beta)}\left(\delta_{\pi(i),\pi'(i)}+\frac{1}{x_\alpha-x_i}\frac{1}{y_\beta-y_{\pi(i)}}\right)
\end{aligned}$$

$(2-2)$

commute together for all α, β*. They also commute with* $\mathcal{M}(\vec{x}, \vec{y}, \xi, \eta)$ *and with* $\mathcal{A}(\vec{x}, \vec{y})$*.*

One has:
$$\mathcal{M}^{(R)}(\vec{x}, \vec{y}, \xi, \eta) = \mathbf{1} + \sum_{\alpha,\beta} \frac{1}{(\xi - x_\alpha)(\eta - y_\beta)}\, \mathcal{A}^{\alpha,\beta}(\vec{x}, \vec{y}) \tag{2-3}$$

References

[1] D. Berenstein, J. Maldacena, H. Nastase "Strings in flat space and pp waves from $\mathcal{N} = 4$ Super Yang Mills", JHEP 0204 (2002) 013.

[2] M. Bergère, "Biorthogonal polynomials for potentials of two variables and external sources at the denominator", hep-th/0404126.

[3] M. Bergère, "Biorthogonal polynomials for potentials of two variables and external sources at the denominator", 4th Eurogrid meeting, Les Houches (march 2004), and Capri meeting (sept 2004), http://www.na.infn.it/congr/Bohr/.

[4] M. Bertola, "Bilinear semi–classical moment functionals and their integral representation", *J. App. Theory* **121** (2003) 71 - 99, math.CA/0205160

[5] "M. Bertola, B.Eynard, "The PDEs of biorthogonal polynomials arising in the 2-matrix model", SPHT T03/139, nlin.SI/0311033.

[6] M. Bertola, B.Eynard, "Mixed Correlation Functions of the Two-Matrix Model" , SPHT T03/028, CRM-2961 (2003). *J. Phys.* **A36** (2003) 7733-7750. xxx, hep-th/0303161.

[7] M. Bertola, B. Eynard, J. Harnad, "Semiclassical orthogonal polynomials, matrix models and isomonodromic tau functions", SPHT T04/019. CRM-3169 (2004), xxx, nlin.SI/0410043,

[8] M. Bertola, B. Eynard, J. Harnad, "Differential systems for biorthogonal polynomials appearing in 2–matrix models and the associated Riemann–Hilbert problem", *Comm. Math. Phys.* **243** (2003) 193-240, nlin.SI/0208002.

[9] M. Bertola, B. Eynard, J. Harnad, "Duality, Biorthogonal Polynomials and Multi–Matrix Models", Commun. Math. Phys. **229** (2002), 73–120.

[10] P. Di Francesco, P. Ginsparg, J. Zinn-Justin, "2D Gravity and Random Matrices", *Phys. Rep.* **254**, 1 (1995).

[11] J.J. Duistermaat, G.J. Heckman, *Inv. Math.* **69** (1982) 259.

[12] B. Eynard, "Eigenvalue distribution of large random matrices, from one matrix to several coupled matrices", *Nuc. Phys.* **B 506**,3 633-664 (1997).

[13] B. Eynard, "A short note about Morozov's formula", SPHT T04/077. xxx, math-ph/0406063, ccsd-00002150.

[14] B. Eynard, "Master loop equations, free energy and correlations for the chain of matrices", *JHEP11(2003)* **018**, xxx, hep-th/0309036.

[15] B. Eynard, "Large N expansion of the 2-matrix model", *JHEP 01 (2003)* **051**, xxx, hep-th/0210047.

[16] B. Eynard, M.L. Mehta, "Matrices coupled in a chain: eigenvalue correlations", *J. Phys. A: Math. Gen.* **31**, 4449 (1998), cond-mat/9710230.

[17] W. Fulton, J. Harris, "Representation theory", Series Graduate Texts in Mathematics, vol 129, Springer, 1991.

[18] C. Itzykson and J.B. Zuber, "The planar approximation (II)", *J. Math. Phys.* **21**, 411 (1980).

[19] V.A. Kazakov, "Ising model on a dynamical planar random lattice: exact solution", *Phys. Lett.* **A 119** (1986) 140.

[20] A. W. Knapp, "Representation theory of semisimple groups", Princeton University Press, Princeton, New Jersey, USA, 1986.

[21] M.L. Mehta, "A method of integration over matrix variables", *Commun. Math. Phys.* **79**, 327 (1981).

[22] M.L. Mehta, *Random Matrices*, 2nd edition, (Academic Press, New York, 1991).

[23] A. Morozov, "Pair correlator in the Itzykson–Zuber Integral", *Modern Phys. Lett.* **A 7**, no. 37 3503–3507 (1992).

[24] S. L. Shatashvili, "Correlation Functions in the Itzykson–Zuber Model", *Comm. Math. Phys.* **154** 421–432 (1993).

[25] M. Staudacher, "Combinatorial Solution of the Two–Matrix Model", *Phys. Lett.* B **305** 332–338 (1993).

[26] J. Ginibre, "Statistical ensembles of complex, quaternion and real matrices", *J. Math. Phys.* 6 (1965) 440.

[27] I.K.Kostov, I.Krichever, M.Mineev-Weinstein, P.Wiegmann, A.Zabrodin, "τ-function for analytic curves", Random matrices and their applications, MSRI publications,vol.40, 285,2001, Cambridge University Press, hep-th/0005259.

[28] A.Zabrodin, "Matrix models and growth processes: from viscous flows to the quantum Hall effect", lectures given at the School "Applications of Random Matrices in Physics", Les Houches, June 2004, ITEP-TH-80/04, hep-th/0412219.